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POWERS FROM PRODUCTS OF CONSECUTIVE TERMS IN ARITHMETIC PROGRESSION

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Dedicated to Professor R. Tijdeman on the occasion of his sixtieth birthday

1. *Introduction*

A celebrated theorem of Erdős and Selfridge [**14**] states that the product of consecutive positive integers is never a perfect power. A more recent and equally appealing result is one of Darmon and Merel [**11**] who proved an old conjecture of Dénes to the effect that there do not exist three consecutive nth powers in arithmetic progression, provided $n \geqslant 3$. One common generalization of these problems is to ask whether it is possible to have a product of consecutive terms in arithmetic progression equal to a perfect power. In general, the answer to this question is 'yes', as the Diophantine equation

$$n(n+d)\dots(n+(k-1)d) = y^l, \quad \text{for } k \geqslant 3 \text{ and } l \geqslant 2, \tag{1}$$

may have infinitely many solutions in positive integers n, d, k, y and l if either the integers n and d have suitable common factors (as in the example $9 \cdot 18 \cdot 27 \cdot 36 = 54^3$), or $(k,l) = (3,2)$ and $\gcd(n,d) = 1$ (for example, $1 \cdot 25 \cdot 49 = 35^2$). If, however, we restrict our attention to progressions with

$$\gcd(n,d) = 1, \quad k \geqslant 3, \quad l \geqslant 2, \quad (k,l) \neq (3,2), \tag{2}$$

a number of special finiteness results are available in the literature. Euler (see for example [**13**]) showed that then (1) has no solutions if $(k,l) = (3,3)$ or $(4,2)$; a similar statement was obtained by Obláth [**26**, **27**] for the cases $(k,l) = (3,4), (3,5)$ or $(5,2)$. It has been conjectured by Erdős (as noted in [**37**]; see also Darmon and Granville [**10**]) that (1) (with (2)) has, in fact, no solutions whatsoever. This conjecture has been recently established by Győry [**18**] for $k = 3$ (and $l \geqslant 3$ arbitrary) and by Győry, Hajdu and Saradha [**19**], in case $k = 4$ or 5. Unfortunately, the arguments of [**19**] are invalid if $l = 3$; we correct these in § 5 of this paper.

In general, however, it appears to be a very hard problem to prove even that the number of solutions to (1), with (2), is finite. As a rough indication of its depth, this does not seem to be a consequence of the ABC Conjecture of Masser and Oesterlé, unless we further assume that $l \geqslant 4$; see Theorem 7 of [**19**]. Further work in this direction, under restrictive hypotheses, includes that of Marszalek [**23**] (in case d is fixed), Shorey and Tijdeman [**37**] (if l and the number of prime divisors of d is

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fixed) and Darmon and Granville [**10**] (if both k and l are fixed). For a broader sample of the abundant literature in this area, the reader may wish to consult the survey articles of Tijdeman [**42**] and Shorey [**35**, **36**].

In this paper, we will address the problem of establishing finiteness results for equation (1), under the sole assumption that k is fixed. One of the principal results of this paper is an extension of the aforementioned work of Győry [**18**] and Győry, Hajdu and Saradha [**19**] to $k \leqslant 11$ (with a requisite correction of the latter work, in case $l = 3$).

THEOREM 1.1. *The product of k consecutive terms in a coprime positive arithmetic progression with $4 \leqslant k \leqslant 11$ can never be a perfect power.*

By coprime progression, we mean one of the form

$$n, n + d, \ldots, n + (k - 1)d$$

with $\gcd(n, d) = 1$. We should emphasize that this does not follow as a mere computational sharpening of the approach utilized in [**18**] or [**19**], but instead necessitates the introduction of fundamentally new ideas. Indeed, the principal novelty of this paper is the combination of a new approach for solving ternary Diophantine equations under additional arithmetic assumptions, via Frey curves and modular Galois representations, with classical (and not so classical!) results on lower degree equations representing curves of small (positive) genus. Further, for the most part, our results do not follow from straightforward application of the modularity of Galois representations attached to Frey curves, but instead require additional understanding of the reduction types of these curves at certain small primes.

Theorem 1.1 is, in fact, an immediate consequence of a more general result. Before we state this, let us introduce some notation. Define, for an integer m with $|m| > 1$, $P(m)$ and $\omega(m)$ to be the largest prime dividing m and the number of distinct prime divisors of m, respectively (where we take $P(\pm 1) = 1$ and $\omega(\pm 1) = 0$). Further, let us write

$$\Pi\,(i_1, i_2, \ldots, i_t) = (n + i_1 d)(n + i_2 d) \ldots (n + i_t d) \tag{3}$$

and

$$\Pi_k = \Pi(0, 1, 2, \ldots, k - 1) = n(n + d)(n + 2d) \ldots (n + (k - 1)d). \tag{4}$$

With these definitions, we have the following theorem.

THEOREM 1.2. *Suppose that k and l are integers with $3 \leqslant k \leqslant 11$, $l \geqslant 2$ prime, and $(k, l) \neq (3, 2)$, and that n and d are coprime integers with $d > 0$. If, further, b and y are non-zero integers with $P(b) \leqslant P_{k,l}$ where $P_{k,l}$ is as shown in Table* 1*, then the only solutions to the Diophantine equation*

$$\Pi = \Pi_k = by^l \tag{5}$$

are with (n, d, k) in the following list:

$$(-9, 2, 9), (-9, 2, 10), (-9, 5, 4), (-7, 2, 8), (-7, 2, 9), (-6, 1, 6), (-6, 5, 4),$$
$$(-5, 2, 6), (-4, 1, 4), (-4, 3, 3), (-3, 2, 4), (-2, 3, 3), (1, 1, 4), (1, 1, 6).$$

TABLE 1. $P_{k,l}$ *for* $3 \leqslant k \leqslant 11$ *and* $l \geqslant 2$.

k	$l=2$	$l=3$	$l=5$	$l \geqslant 7$
3	–	2	2	2
4	2	3	2	2
5	3	3	3	2
6	5	5	5	2
7	5	5	5	3
8	5	5	5	3
9	5	5	5	3
10	5	5	5	3
11	5	5	5	5

For $k = 3$, this theorem was proved in [**18**]. Our Theorem 1.2 sharpens and generalizes the corresponding results of [**19**], which treated the cases $k = 4$ and 5 (with $l \neq 3$). Note that the upper bound on $P(b)$ in the above theorem may be replaced in all cases by the slightly stronger but simpler bound

$$P(b) < \max\{3, k/2\}, \tag{6}$$

leading to a cleaner but weaker theorem. Further, in cases $(k, l) = (4, 2)$ and $(3, 3)$, the result is best possible (in the sense that $P_{k,l}$ cannot be replaced by a larger value). This is almost certainly not true for other values of (k, l).

It is a routine matter to extend Theorem 1.2 to arbitrary (that is, not necessarily prime) values of l. For $(k, l) = (3, 4)$, equation (5) has no solutions with (6); cf. Theorem 8 of [**19**]. For all other pairs (k, l) under consideration, Theorem 1.2 yields the following result.

COROLLARY 1.3. *Suppose that* n, d *and* k *are as in Theorem* 1.2, *and that* $l \geqslant 2$ *is an integer with* $(k, l) \neq (3, 2)$. *If, further,* b *and* y *are non-zero integers with* (6), *then the only solutions to equation* (5) *are with* (n, d, k) *in the following list:*

$$(-9, 2, 9), (-9, 2, 10), (-9, 5, 4), (-7, 2, 8), (-7, 2, 9),$$
$$(-6, 5, 4), (-5, 2, 6), (-4, 3, 3), (-3, 2, 4), (-2, 3, 3).$$

Note that knowing the values of the unknowns on the left-hand side of (5), one can easily determine all the solutions (n, d, k, b, y, l) to (5).

In the special case $d = 1$, the set of solutions of equation (5), for $k \geqslant 2$ fixed, has been described in [**17**, **20**, **31**], under less restrictive assumptions upon b. For further partial results on (5), we refer again to the survey papers [**18**, **35**, **36**, **42**].

For fixed values of $k \geqslant 3$ and $l \geqslant 2$ with $k+l > 6$, equation (5) has at most finitely many solutions in positive integers (n, d, b, y) with $\gcd(n, d) = 1$ and $P(b) \leqslant k$; see Theorem 6 of [**19**].

If we turn our attention to $k > 11$, we may prove a number of results of a similar flavour to Theorem 1.2, only with a corresponding loss of precision. If k is slightly larger than 11, we have the following theorem.

THEOREM 1.4. *If* $12 \leqslant k \leqslant 82$, *then there are at most finitely many non-zero integers* n, d, l, b *and* y *with* $\gcd(n, d) = 1$, $l \geqslant 2$ *and satisfying* (5), *with* $P(b) < k/2$.

Moreover, for all such solutions to (5), *we have*

$$\log P(l) < 3^k.$$

For arbitrary values of k, we may deduce finiteness results for equations (1) and (5), only under certain arithmetic assumptions. Write

$$D_k = \prod_{k/2 \leqslant p < k} p \tag{7}$$

where the product is over prime p.

THEOREM 1.5. *If $k \geqslant 4$ is fixed, then the Diophantine equation* (5) *has at most finitely many solutions in positive integers n, d, b, y and l with*

$$\gcd(n, d) = 1, \quad y > 1, \quad l > 1, \quad P(b) < k/2 \quad \text{and} \quad d \not\equiv 0 \pmod{D_k}.$$

For each such solution, we necessarily have $\log P(l) < 3^k$.

A corollary of this which yields a finiteness result for (1), provided k is suitably large (relative to the number of prime divisors of d), is the following.

COROLLARY 1.6. *Let D be a positive integer and suppose that k is a fixed integer satisfying*

$$k \geqslant \begin{cases} 4 & \text{if } D \in \{1, 2\}, \\ 6D \log D & \text{if } D \geqslant 3. \end{cases} \tag{8}$$

Then the Diophantine equation (5) *has at most finitely many solutions in positive integers n, d, b, y and l with*

$$\gcd(n, d) = 1, \quad y > 1, \quad l > 1, \quad \omega(d) \leqslant D, \quad \text{and} \quad P(b) < k/2.$$

We remark that a sharp version of this result, in the special case $l = 2$ and $b = D = 1$, was recently obtained by Saradha and Shorey [**33**].

Finally, we mention an application of Theorem 1.2 to a family of superelliptic equations first studied by Sander [**30**]. Specifically, let us consider equations of the form

$$x(x + 1) \dots (x + k - 1) = \pm 2^\alpha z^l \tag{9}$$

where x and z are rational numbers with $z \geqslant 0$, and k, l and α are integers with $k, l \geqslant 2$ and $-l < \alpha < l$. If $-l < \alpha < 0$, by replacing α and z in (9) with $l + \alpha$ and $z/2$, respectively, we may restrict ourselves to the case where α is non-negative.

If x and z are further assumed to be integers and $\alpha = 0$, then, by the result of Erdős and Selfridge [**14**], we find that the only solutions to (9) are with $z = 0$. Since these are also solutions of (9) for each α, we will henceforth refer to them as *trivial*; in what follows, we shall consider only non-trivial solutions. Let us return to the more general situation when $x, z \in \mathbb{Q}$. By putting $x = n/d$ and $z = y/u$ with integers n, d, y and u such that $\gcd(n, d) = \gcd(y, u) = 1$, $d > 0$, $y \geqslant 0$ and $u > 0$, we see that (9) reduces to equation (5) with $P(b) \leqslant 2$ and (by comparing denominators) satisfying the additional constraint that $u^l = 2^\gamma d^k$ for some non-negative integer γ. An almost immediate consequence of Theorem 1.2 is the following.

COROLLARY 1.7. *Let $2 \leqslant k \leqslant 11$ and $l \geqslant 2$ with $(k,l) \neq (2,2)$ (and, if $\alpha > 0$, $(k,l) \neq (2,4)$). Then the only non-trivial solutions of (9) with $0 \leqslant \alpha < l$ are those (x,k) in the following list:*

$$(-9/2,9),(-9/2,10),(-7/2,8),(-7/2,9),(-5/2,6),(-2,2),$$
$$(-3/2,4),(-4/3,3),(-2/3,3),(-1/2,2),(1,2).$$

This result follows easily from Theorem 1.2; the reader is directed to [**19**] for the necessary arguments. Indeed, in [**19**], our Corollary 1.7 is established for $l \geqslant 4$, $k = 3,4$ and, if $\alpha = 0$ and $k = 5$. If $2 \leqslant k \leqslant 4$, $l > 2$ and $\alpha = 0$, Sander [**30**] completely solved equation (9) and noted that, for $(k,l) = (2,2)$, there are, in fact, infinitely many solutions. We remark, however, that the solutions listed in Corollary 1.7 for $k = 3$ and 4 are missing from Sander's result. Further, as discussed in [**19**], the assumption $(k,l) \neq (2,4)$ (if $\alpha > 0$) is necessary, since, in that case, equation (9) has, again, infinitely many solutions.

The structure of this paper is as follows. In the second section, we will indicate how the problem of solving equation (5) may be translated to a question of determining solutions to ternary Diophantine equations. In §§ 3–6, we prove Theorem 1.2 for, respectively, prime $l \geqslant 7$, $l = 2$, $l = 3$ and $l = 5$. In many cases, for $l = 2$ or 3, the problem may be reduced to one of finding the torsion points on certain rank 0 elliptic curves $E/\mathbb{Q}$. In a number of situations, however, this approach proves inadequate to deduce the desired result. We instead turn to recent explicit Chabauty techniques due to Bruin and Flynn [**6**]; we encounter some interesting variations between the cases with $l = 2$ and those with $l = 3$. If $l = 5$, we depend on either classical results of Dirichlet, Lebesgue, Maillet (cf. [**13**]), Dénes [**12**] and Győry [**16**] on generalized Fermat equations of the shape $X^l + Y^l = CZ^l$, or recent work of Kraus [**21**]. For $l \geqslant 7$, we apply recent results of the first author and Skinner [**1**], together with some refinements of these techniques; our proofs are based upon Frey curves and the theory of Galois representations and modular forms. Section 7 is devoted to the proof of Theorem 1.5. Finally, we conclude the paper by considering values of k with $12 \leqslant k \leqslant 82$.

2. *The transition to ternary equations*

For virtually every argument in this paper, we will rely heavily on the fact that a 'non-trivial' solution to (5) implies a number of similar solutions to related ternary Diophantine equations which we may, if all goes well, be able to treat with the various tools at our disposal. The only situation where we will not follow this approach is in § 4 (that is, when $l = 2$). From equation (5) and the fact that $\gcd(n,d) = 1$, we may write

$$n + id = b_i y_i^l \quad \text{for } 0 \leqslant i \leqslant k-1, \tag{10}$$

where b_i and y_i are integers with $P(b_i) < k$. We note that, in terms of b_i, such a representation is not necessarily unique. We will thus assume, unless otherwise stated, that each b_i is lth power free and, if l is odd, positive.

Let us first observe that any three of the linear forms $n + id$, for $0 \leqslant i \leqslant k-1$, are linearly dependent. In particular, given distinct integers $0 \leqslant q,r,s \leqslant k-1$, we may find relatively prime non-zero integers λ_q, λ_r, λ_s, for which

$$\lambda_q(n+qd) + \lambda_r(n+rd) = \lambda_s(n+sd). \tag{11}$$

It follows from (10) that, writing

$$A = \lambda_q b_q, \quad B = \lambda_r b_r, \quad C = \lambda_s b_s, \quad \text{and} \quad (u, v, z) = (y_q, y_r, y_s),$$

we have

$$Au^l + Bv^l = Cz^l, \tag{12}$$

where it is straightforward to show that $P(ABC) < k$. This is a ternary Diophantine equation of signature (l, l, l). In case $l = 3, 5$ and, sometimes, $l \geqslant 7$, we will prove Theorem 1.2 through analysis of such equations. In the sequel, we will employ the shorthand $[q, r, s]$ to refer to an identity of the form (11) (and hence a corresponding equation (12)), because given distinct integers q, r and s, coprime non-zero integers λ_q, λ_r and λ_s satisfying (11) are unique up to sign.

A second approach to deriving ternary equations from a solution to (5) proves to be particularly useful for larger values of (prime) l. If p, q, r and s are integers with

$$0 \leqslant p < q \leqslant r < s \leqslant k - 1 \quad \text{and} \quad p + s = q + r,$$

then we may observe that

$$(n + qd)(n + rd) - (n + pd)(n + sd) = (qr - ps)d^2 \neq 0. \tag{13}$$

It follows that identity (13) implies (non-trivial) solutions to Diophantine equations of the form

$$Au^l + Bv^l = Cz^2 \tag{14}$$

with $P(AB) < k$, for each quadruple $\{p, q, r, s\}$. This is a ternary Diophantine equation of signature $(l, l, 2)$. Henceforth, we will use the shorthand $\{p, q, r, s\}$ to refer to an identity of the form (13).

Our arguments will rely upon the fact that a triple $[q, r, s]$ or quadruple $\{p, q, r, s\}$ can always be chosen such that the resulting equation (12) or (14) is one that we may treat with techniques from the theory of Galois representations and modular forms, or, perhaps, with a more classical approach. In essence, once we have established certain results on the equations (12) and (14), as we shall see, this can be regarded as a purely combinatorial problem.

3. *Proof of Theorem* 1.2 *in case* $l \geqslant 7$

We will primarily treat equation (5) with prime exponent $l \geqslant 7$ by reducing the problem to one of determining the solvability of equations of the shape (14). For a more detailed discussion of these matters, the reader is directed to [**1**, **11**, **22**, **25**]. We begin by cataloguing the required results on such ternary equations.

PROPOSITION 3.1. *Let $l \geqslant 7$ be prime, α and β be non-negative integers, and let A and B be coprime non-zero integers. Then the following Diophantine equations*

have no solutions in non-zero coprime integers (x, y, z) *with* $xy \neq \pm 1$*:*

$$x^l + 2^\alpha y^l = 3^\beta z^2, \quad \alpha \neq 1, \tag{15}$$

$$x^l + 2^\alpha y^l = z^2 \quad \text{with } p \mid xy \text{ for } p \in \{3, 5, 7\}, \tag{16}$$

$$x^l + 2^\alpha y^l = 3z^2 \quad \text{with } p \mid xy \text{ for } p \in \{5, 7\}, \tag{17}$$

$$x^l + y^l = Dz^2, \quad D \in \{2, 6\}, \tag{18}$$

$$x^l + 3^\alpha y^l = 2z^2 \quad \text{with } p \mid xy \text{ for } p \in \{5, 7\}, \quad l \geqslant 11, \tag{19}$$

$$x^l + 5^\alpha y^l = 2z^2 \quad \text{with } l \geqslant 11 \text{ if } \alpha > 0, \tag{20}$$

$$Ax^l + By^l = z^2, \quad AB = 2^\alpha p^\beta, \quad \alpha \geqslant 6, \quad p \in \{3, 5, 13\}, \tag{21}$$

$$Ax^l + By^l = z^2, \quad AB = 2^\alpha p^\beta, \quad \alpha \neq 1, \quad p \in \{11, 19\}, \tag{22}$$

$$Ax^l + By^l = z^2, \quad P(AB) \leqslant 3, \quad \text{with } p \mid xy \text{ for } p \in \{5, 7\}, \tag{23}$$

$$Ax^l + By^l = z^2, \quad P(AB) \leqslant 5, \quad \text{with } 7 \mid xy \text{ and } l \geqslant 11. \tag{24}$$

In each instance where we refer to a prime p, we further suppose that the exponent $l > p$.

Proof. We begin by noting that the stated results for equations (15), (18), (20) and (22) are, essentially, available in a paper by Bennett and Skinner [**1**]. The cases of equation (21) with $p = 3$ or 5, and $\beta \geqslant 1$, while not all explicitly treated in [**1**], follow immediately from the arguments of that paper, upon noting that the modular curves $X_0(N)$ have genus 0 for all N dividing 6 or 10.

For the remaining equations, we will begin by employing the approach of [**1**]. Specifically, to a putative non-trivial solution of one of the preceding equations, we associate a Frey curve $E/\mathbb{Q}$ (see [**1**] for details), with corresponding mod l Galois representation

$$\rho_l^E : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_l)$$

on the l-torsion $E[l]$ of E. Via Lemmata 3.2 and 3.3 of [**1**], this representation arises from a cuspidal newform f of weight 2 and trivial Nebentypus character. The level N of this newform may be shown to satisfy

$$N \in \{20, 24, 30, 40, 96, 120, 128, 160, 384, 480, 640, 768, 1152, 1920\}$$

(for example, a non-trivial solution to (16) with $\alpha = 1$ and x and y odd necessarily leads to a newform of level 128; for details, the reader is directed to Lemma 3.2 of [**1**]). The assumption that $p \mid xy$ for $p \in \{3, 5, 7\}$ implies, if p is coprime to lN, that

$$\mathrm{trace}\, \rho_l^E\, (\mathrm{Frob}_p) = \pm(p + 1).$$

It follows, if f has Fourier coefficients a_n in a number field K_f, that

$$\mathrm{Norm}_{K_f/\mathbb{Q}}\, (a_p \pm (p + 1)) \equiv 0 \pmod{l}. \tag{25}$$

Using William Stein's 'Modular Forms Database' [**38**], we find a_p, with $p \in \{3, 5, 7\}$, for each newform at the levels N of interest, provided p is coprime to N. In most cases the corresponding Fourier coefficients are even integers: from the Weil bounds, $a_3 \in \{0, \pm 2\}$ (if $3 \nmid N$), $a_5 \in \{0, \pm 2, \pm 4\}$ (if 5 is coprime to N) and $a_7 \in \{0, \pm 2, \pm 4\}$ (if 7 fails to divide N). Congruence (25) thus implies a contradiction for these forms. The only forms f encountered with $K_f \neq \mathbb{Q}$ are (in Stein's notation) form 3 at level 160, forms 9–12 at level 640, forms 9–12 at level

768 and forms 25–28 at level 1920. In the case of form 3, $N = 160$, we find that $a_7 = \pm 2\sqrt{2}$ and so $2\sqrt{2} \equiv \pm 8 \pmod{\mathcal{P}}$ for some prime $\mathcal{P}$ lying over l. It follows that $l \mid 56$ and so $l = 7$. Similarly, form 9 at level 672 has $a_7 = -\vartheta - 2$ where $\vartheta^2 + 2\vartheta - 4 = 0$. From $a_7 \equiv \pm 8 \pmod{\mathcal{P}}$ we thus have $\vartheta \equiv 6 \pmod{\mathcal{P}}$ (whereby $l = 11$) or $\vartheta \equiv -10 \pmod{\mathcal{P}}$ (whence $l = 19$). On the other hand, $a_3 = \vartheta$ and hence, from the Weil bounds, $\vartheta \equiv 0, \pm 2, \pm 4 \pmod{\mathcal{P}}$, a contradiction in each case. Arguing in a like fashion for the remaining forms completes the proof. □

We will also need a result on equations of signature (l, l, l). Specifically, we apply the following.

PROPOSITION 3.2. *Let $l \geqslant 3$ and $\alpha \geqslant 0$ be integers. Then the Diophantine equation*

$$X^l + Y^l = 2^\alpha Z^l \tag{26}$$

has no solutions in coprime non-zero integers X, Y and Z with $XYZ \neq \pm 1$.

Proof. This is essentially due to Wiles [**43**] (in case $l \mid \alpha$), Darmon and Merel [**11**] (if $\alpha \equiv 1 \pmod{l}$) and Ribet [**28**] (in the remaining cases for $l \geqslant 5$ prime); see also Győry [**18**]. □

Let us begin the proof of Theorem 1.2. For the remainder of this section, we will suppose that there exists a solution to equation (5) in non-zero integers n, d, k, y, l and b with n and $d > 0$ coprime, $3 \leqslant k \leqslant 11$, and $l \geqslant 7$ prime. We suppose further that b satisfies (6). We treat each value $3 \leqslant k \leqslant 11$ in turn.

3.1. *The case $k = 3$*

If $k = 3$, the identity $\{0, 1, 1, 2\}$ yields solutions to an equation of the shape (15) with $\beta = 0$ and $\alpha = 0$ (if Π is odd) or $\alpha \geqslant 2$ (if Π is even). By Proposition 3.1, after a modicum of work, we obtain the solutions $(n, d, k) = (-4, 3, 3)$ and $(-2, 3, 3)$ listed in the statement of Theorem 1.2.

3.2. *The case $k = 4$*

If n is coprime to 3, we may use the same identity as for $k = 3$ to deduce that there is no solution to (5). If $3 \mid n$, then $\{0, 1, 2, 3\}$ gives an equation of type (18) with $D = 2$ (if Π is odd), and one of the form (16) with $p = 3$ (if Π is even). In either case, we infer from Proposition 3.1 that equation (5) has no solution.

3.3. *The case $k = 5$*

Considering the product of the first or the last four terms of Π, according as $3 \mid n$, or not, we may reduce this to the preceding case and reach the desired conclusion.

3.4. *The case $k = 6$*

If $k = 6$ and 5 fails to divide n, then we may apply what we have for the case $k = 4$ to the product of the first, middle or last four terms of Π, to find that there is no solution to (5). Similarly, if $3 \nmid n(n + 5d)$, the middle four terms lead to a contradiction. Thus we may suppose that $5 \mid n$, and, by symmetry, that also $3 \mid n$.

Considering the identity $\{0,1,4,5\}$ (if Π is odd) or $\{0,2,3,5\}$ (if Π is even), we obtain an equation of the shape (23) with $p=5$. We can thus apply Proposition 3.1 to conclude that (5) has no solution with $k=6$ and $l \geqslant 7$ prime.

3.5. *The case $k=7$*

Next, let $k=7$. If $5 \nmid n(n+d)$, then we may apply $\{1,2,4,5\}$ (if $3 \mid n$) or $\{0,3,3,6\}$ (if $3 \nmid n$). These lead to equations of type (15). Next, suppose that $5 \mid n(n+d)$; by symmetry, we may assume $5 \mid n$. Suppose first that $6 \mid \Pi$, and consider the identity $\{0,2,3,5\}$. If $3 \mid n+d$, we are led to an equation of the shape (16) or (17), with $p=5$. On the other hand, if $3 \mid n(n+2d)$, then the same identity induces an equation of the form (23), again with $p=5$.

Assume now that $6 \nmid \Pi$, and consider $\{0,1,4,5\}$. If $\gcd(\Pi,6)=3$, this identity gives equation (23) with $p=5$. If, however, $\gcd(\Pi,6)=2$, then the same identity leads either to (16) with $p=5$ or to (18), with $D=2$. Finally, if $\gcd(\Pi,6)=1$, then again employing the identity $\{0,1,4,5\}$, we find a solution to (15) with $\alpha=\beta=0$. In all cases, we conclude from Proposition 3.1 that (5) has no solution, in the situation under consideration.

3.6. *A diversion*

In case $k \geqslant 8$, in a number of instances, Proposition 3.1 enables us to prove our statement only for $l \geqslant 11$ prime. We are thus forced to deal with the exponent $l=7$ separately. As we shall observe, in each case where we encounter difficulties for $l=7$, there are precisely two distinct factors in Π which are divisible by 7. By our assumptions, $7 \mid \nu_7(\Pi)$ where, here and henceforth, we write $\nu_p(m)$ for the largest integer t such that p^t divides a non-zero integer m. It follows that one of these two factors is necessarily divisible by 7^2. We will use the following argument to finish the proof in this case.

Choose three factors $n+qd$, $n+rd$ and $n+sd$ of Π, such that one of them, $n+qd$ say, is divisible by 7^2, but 7 fails to divide $(n+rd)(n+sd)$. The identity $[q,r,s]$ thus yields

$$\lambda_r b_r y_r^7 \equiv \lambda_s b_s y_s^7 \pmod{7^2},$$

whence, upon taking sixth powers, it follows that

$$u^6 \equiv v^6 \pmod{7^2}, \tag{27}$$

where $u=\lambda_r b_r$ and $v=\lambda_s b_s$. If we choose $n+qd$, $n+rd$ and $n+sd$ appropriately, then we can use the fact that, for $a \equiv uv^{-1} \pmod{7^2}$,

$$a^6 \equiv 1 \pmod{7^2} \iff a \equiv \pm 1, \pm 18, \pm 19 \pmod{7^2} \tag{28}$$

to obtain a contradiction, thereby verifying that (5) has no solution in the case in question.

3.7. *The case $k=8$*

Let us return to our proof. Suppose $k=8$. If $7 \nmid n$, then we may reduce to the preceding case by considering the first or last seven terms of Π. Suppose, then, that $7 \mid n$. Notice that if $\gcd(\Pi,15)=1$, then we may apply our results with $k=6$ to the middle six terms of Π to conclude that (5) has no solution. If $5 \nmid \Pi$, it therefore

follows that $3 \mid \Pi$. If $3 \mid n$ or $3 \mid n+d$, using $\{1,2,4,5\}$ or $\{2,3,5,6\}$ respectively, we are led to an equation of the shape (15) with $\beta = 1$, contradicting Proposition 3.1. If $3 \mid n+2d$, then the identity $\{0,1,6,7\}$ gives rise to an equation of the form (18) with $D = 6$, if Π is odd, and of the form

$$x^l + 2^\alpha y^l = 3z^2, \tag{29}$$

if Π is even. We may apply Proposition 3.1 again, unless $\alpha = 1$, that is, unless $\nu_2(n+id) = 2$ for one of $i = 0,1,6,7$. If this last condition occurs, it follows that $\nu_2(n+jd) \geqslant 3$ for one of $j = 2,3,4,5$. For this j, the identity $\{j-1, j, j, j+1\}$ leads to an equation of the form (21) with $p = 3$. By Proposition 3.1, we infer that (5) has no solution in this case.

We may thus suppose that $5 \mid \Pi$. If $3 \nmid \Pi$, then we may apply our results obtained for $k = 3$ to $\Pi(i, i+1, i+2)$ with an appropriate $i = 1, 3$ or 4 to conclude that there is no solution in this case. We may therefore assume that $15 \mid \Pi$. Further, if $5 \mid (n+3d)(n+4d)$, we can argue as previously to obtain a contradiction. Hence we may suppose that $5 \mid n(n+d)(n+2d)$. Assume first that $5 \mid n+d$. If Π is odd, then the identity $\{1,2,5,6\}$ leads to (23) with $p = 5$ and so, via Proposition 3.1, a contradiction. If Π is even, then we consider the identity $\{1,3,4,6\}$. If $3 \mid n+2d$, we are led to an equation of the form (17) with $p = 5$. On the other hand, if $3 \mid n(n+d)$, then we find a non-trivial solution to (23) with $p = 5$. In either case, we contradict Proposition 3.1.

To complete the proof of Theorem 1.2, in case $k = 8$, we may thus, by symmetry, suppose that $5 \mid n$. We divide our proof into two parts. First suppose that $l \geqslant 11$ prime.

We begin with the case where $3 \mid n$. Necessarily, one of n, $n+3d$ or $n+6d$ is divisible by 9. If $9 \mid n$, then $\{1,3,4,6\}$ gives rise to an equation of the form (18) with $D = 2$, at least provided Π is odd. When Π is even, the identity $\{0,2,5,7\}$ yields (24) and hence a contradiction. If $9 \mid n+3d$, $\{0,1,6,7\}$ leads to (20), if Π is odd. If Π is even, from the same identity we have (24). By Proposition 3.1, in each case, we conclude that there is no solution to (5). Finally, if $9 \mid n+6d$, then the identity $\{0,3,4,7\}$ provides either (20) or (24). In both cases, we have a contradiction, at least for $l \geqslant 11$ prime.

We argue in a similar fashion if $3 \mid n+d$ or $3 \mid n+2d$. In the first of these cases, one of the identities $\{0,3,4,7\}$, $\{0,1,6,7\}$, $\{1,3,4,6\}$ or $\{0,2,5,7\}$, necessarily implies solutions to either (20) or (24). In the second, either $\{1,3,4,6\}$ yields a solution to (18) with $D = 6$, or $\{0,2,5,7\}$ provides one to equation (24). By Proposition 3.1, we thus derive a contradiction, in all cases, for $l \geqslant 11$ prime.

Now suppose that $l = 7$. We use the argument outlined in § 3.6; that is, we appeal to identities of the form (11), corresponding to triples $[q,r,s]$.

Assume first that, together with $5 \mid n$, we have $3 \mid n$. Since, necessarily, either n or $n+7d$ is divisible by 7^2, we distinguish two cases. Suppose first that $7^2 \mid n$, and consider the identity $[0,2,4]$. This implies a congruence of the form

$$\left(2^{\nu_2(b_2)+1}\right)^6 \equiv \left(2^{\nu_2(b_4)}\right)^6 \pmod{7^2},$$

whereby, from (28), $(\nu_2(b_2), \nu_2(b_4)) = (0,1)$ or $(1,2)$. From the identity $\{1,2,2,3\}$, if $\nu_2(n+2d) \geqslant 3$, we derive a non-trivial solution to (21) with $p = 3$, contrary to Proposition 3.1. We conclude, then, that $\nu_2(n+2d) = 1$ (and hence $\nu_2(n+6d) = 1$).

It thus follows, from $[0,3,6]$, that

$$\left(3^{\nu_3(b_3)}\right)^6 \equiv \left(3^{\nu_3(b_6)}\right)^6 \pmod{7^2}$$

and so $\nu_3(b_3) = \nu_3(b_6) = 1$. The identity $[3,4,6]$ thus leads to a non-trivial solution to equation (26), with $n = 7$ and $\alpha = 1$, contradicting Proposition 3.2.

We next suppose that $7^2 \mid n + 7d$. If $2 \nmid \Pi$, then $[1,4,7]$ immediately contradicts (28). If $2 \mid n$, arguing as previously, we find, from $[1,3,7]$, that $\nu_3(b_3) = 4$ and hence $[2,3,7]$ implies that $\nu_2(b_2) = 6$. If, however, $2 \mid n+d$, $[4,6,7]$ gives $\nu_3(b_6) = 6$ whence, from $[3,6,7]$, we have $\nu_2(b_3) = 6$. In either case, $[3,4,7]$ now contradicts (28).

Assume next that $3 \mid n+d$. Suppose first that $7^2 \mid n$. The identity $[0,2,6]$ implies that

$$\left(3 \cdot 2^{\nu_2(b_2)}\right)^6 \equiv \left(2^{\nu_2(b_6)}\right)^6 \pmod{7^2}$$

and so

$$\nu_2(b_6) - \nu_2(b_2) \in \{-4, 3\}. \tag{30}$$

On the other hand, $[0,3,6]$ implies that $\nu_2(b_6) = 1$, contradicting (30) (since we have $\min\{\nu_2(n+2d), \nu_2(n+6d)\} \leqslant 2$).

Next, let $7^2 \mid n + 7d$. In this case, the identity $[2,6,7]$ plays the role of $[0,2,6]$ in the previous situation. We have

$$_2(b_2) - \nu_2(b_6) \in \{-4, 3\}$$

and hence, since $[3,6,7]$ implies that $\nu_2(b_6) = 5$, again a contradiction.

Finally, suppose that $3 \mid n + 2d$. As the situation with Π odd was covered previously for $l = 7$, we need distinguish only two cases. If $2 \mid n$, then $[0,1,3]$ (if $7^2 \mid n$) or $[1,3,7]$ (if $7^2 \mid n + 7d$) each contradict (28). If, however, $2 \mid n + d$, the identities $[0,4,6]$ and $[4,6,7]$ play a like role. This completes the proof of Theorem 1.2 for $k = 8$ and $l \geqslant 7$ prime.

3.8. *The case $k = 9$*

Next, consider $k = 9$. Symmetry allows us to assume that $7 \mid n$, otherwise we can reduce to the preceding situation. We may also assume that $5 \mid n + 3d$, or, by applying our results with $k = 8$ to the first eight terms of Π, again obtain a contradiction. If 3 fails to divide the product Π, then we may use what we have proved already for $k = 3$, via consideration of $\Pi(4,5,6)$, to deduce a contradiction. If $3 \mid n$, then $\{1,2,4,5\}$ yields (15) with $\beta = 1$. Similarly, if $3 \mid n + d$, then $\{3,5,6,8\}$ provides (18) with $D = 6$ if Π is odd, and (17) with $p = 5$ if Π is even. Using Proposition 3.1, we obtain contradictions in either case. If $3 \mid n + 2d$, then the identity $\{0,1,6,7\}$ gives rise to an equation of the shape (18) with $D = 6$ if Π is odd, while $\{3,5,6,8\}$ leads to an equation of the form (23) with $p = 5$ if Π is even. Applying Proposition 3.1 thus completes the proof of Theorem 1.2, in case $k = 9$ and $l \geqslant 7$ prime.

3.9. *The case $k = 10$*

When $k = 10$, we reduce to the preceding case unless either $7 \mid n$ and $5 \mid n + 9d$, or $5 \mid n$ and $7 \mid n + 9d$. By symmetry, we may suppose that the first of these occurs. Then, if $3 \nmid \Pi$, we may apply our result with $k = 3$ for $\Pi(1,2,3)$ to obtain

a contradiction. In case $3 \mid n(n+d)$, $\{2,5,5,8\}$ yields (15) with $\beta = 0$, providing a contradiction by Proposition 3.1. We thus suppose that $3 \mid n+2d$. To complete the proof of Theorem 1.2 in this case, we will utilize Proposition 3.2. Necessarily, precisely one of $n+2d$, $n+5d$ or $n+8d$ is divisible by 9. If $9 \mid n+2d$, the identity $[5,6,8]$ implies a non-trivial solution to (26), contradicting Proposition 3.2. Similarly, if $9 \mid n+5d$ or $9 \mid n+8d$, application of $[2,3,8]$ or $[2,3,5]$, respectively, leads to a contradiction. We conclude, then, that equation (5) has no solution, with $k = 10$ and, again, prime $l \geqslant 7$.

3.10. *The case $k = 11$*

Finally, let $k = 11$. If $5 \nmid \Pi$, then we may apply the results from the preceding case to the first or last ten terms of Π, to obtain a contradiction. If $5 \mid \Pi$, we will, as when $k = 10$, repeatedly appeal to Proposition 3.2 to complete the proof. In what follows, we will assume, via symmetry, that either $7 \nmid \Pi$ or $7 \mid (n+4d)(n+5d)(n+6d)$, or that $7 \mid n(n+d)$. The last case is the only one in which $7 \mid b_i$ for some $0 \leqslant i \leqslant 10$.

Let us begin by supposing that $5 \mid n$. From the identity $\{3,6,6,9\}$, we deduce a solution to (15) unless $3 \mid n$. If $3 \mid n$, then 9 divides exactly one of n, $n+3d$ or $n+6d$. If $9 \mid n$, then $[3,4,6]$ thus implies a (non-trivial) solution to (26), contrary to Proposition 3.2. Similarly, $[6,7,9]$ (if $7 \mid n+d$) and $[6,8,9]$ (in the remaining cases) lead to the same conclusion if $9 \mid n+3d$. Finally, if $9 \mid n+6d$, we may apply $[3,7,9]$ (if $7 \mid n+d$) and $[1,3,9]$ (in the remaining cases) to reach a contradiction.

In case $5 \mid n+id$ for $i = 1, 2$ or 4, we argue similarly. In the first of these cases, either $\{4,7,7,10\}$ (if $7 \mid n+d$) or $\{2,5,5,8\}$ (otherwise) implies that $3 \mid n+d$ (respectively, $3 \mid n+2d$). The identities $[4,5,7]$, $[7,9,10]$ and $[2,4,10]$ (respectively, $[2,3,5]$, $[5,7,8]$ and $[2,4,8]$) thus combine to contradict Proposition 3.2. If $5 \mid n+2d$, $\{3,6,6,9\}$ leads to the conclusion that $3 \mid n$, whereby $[3,4,6]$, $[3,5,9]$ and either $[0,4,6]$ (if $7 \mid n+d$) or $[6,8,9]$ (in all other cases) provide the desired conclusion. If $5 \mid n+4d$, we combine the identities $\{2,5,5,8\}$, $[2,3,5]$, $[5,6,8]$ and $[2,8,10]$ (if $7 \mid n$), or $\{0,3,3,6\}$, $[0,2,3]$, $[3,5,6]$ and $[0,2,6]$ (in all other cases) to obtain a contradiction.

It remains, then, to deal with the possibility that $5 \mid n+3d$. In this situation, we require a somewhat more involved argument. If n is not divisible by 7, then $\{4,7,7,10\}$, together with Proposition 3.1, implies that $3 \mid n+d$, whereby one of $[4,6,7]$, $[7,9,10]$ or $[2,4,10]$ contradicts Proposition 3.2. We may thus suppose that $7 \mid n$. In this case, $\{1,2,4,5\}$ yields a solution to (15) unless $3 \mid (n+d)(n+2d)$. If $3 \mid n+2d$, then $\{0,1,6,7\}$ implies a solution to either (15) or (18) (with $D = 6$), unless

$$\max\{\nu_2(n+id) : i = 0,1,6,7\} = 2. \tag{31}$$

In the latter case, from $\{0,1,6,7\}$, we have a solution to (17) (with $p = 7$) and hence may conclude further that $l = 7$. If $7^2 \mid n$, the identity $[0,1,9]$ implies that

$$\left(9 \cdot 2^{\nu_2(b_1)}\right)^6 \equiv \left(2^{\nu_2(b_9)}\right)^6 \pmod{7^2},$$

contrary to (31). If $7^2 \mid n+7d$, then, from $[1,7,9]$,

$$\left(3 \cdot 2^{\nu_2(b_9)}\right)^6 \equiv \left(2^{\nu_2(b_1)}\right)^6 \pmod{7^2},$$

again contradicting (31).

Finally, if $3 \mid n+d$, from $\{2,5,6,9\}$, we deduce solutions to either (15) or (18) (with $D=6$), unless

$$\max\{\nu_2(n+id) : i = 2,5,6,9\} = 3. \tag{32}$$

In this case, $\{0,1,6,7\}$ implies solutions to equation (24) and so, via Proposition 3.1, we may assume further that $l=7$. If $7^2 \mid n$, then $[0,2,6]$ gives

$$\left(3 \cdot 2^{\nu_2(b_2)}\right)^6 \equiv \left(2^{\nu_2(b_6)}\right)^6 \pmod{7^2},$$

contradicting (32). If $7^2 \mid n+7d$, then $[2,6,7]$ yields

$$\left(5 \cdot 2^{\nu_2(b_6)}\right)^6 \equiv \left(2^{\nu_2(b_2)}\right)^6 \pmod{7^2},$$

again contrary to (32). This completes the proof of Theorem 1.2, in case $l \geqslant 7$ is prime.

4. *Proof of Theorem* 1.2 *in case* $l = 2$

Having disposed of the possibility of equation (5) having solutions with l divisible by a large prime, we are now left with the task of dealing with the primes $l = 2, 3$ and 5. In this section, we treat the first of these cases. For $l=2$ and fixed $k \geqslant 4$, a solution to (5) corresponds to a rational point on one of finitely many hyperelliptic curves. Our argument will essentially rely upon the fact that, with the given restrictions on b, the curves in question may often be shown to cover elliptic curves of rank 0 over $\mathbb{Q}$.

4.1. *The case* $k = 4$

In case $k=4$, we actually deduce a stronger result, which will prove useful for larger values of k.

LEMMA 4.1. *The only solutions in coprime non-zero integers n and d, with $d>0$, and non-zero integer y, to the Diophantine equations*

$$\Pi(0,1,2,3) = by^2, \quad b \in \{\pm 1, \pm 2, \pm 3, 5, -6, 15, -30\}, \tag{33}$$
$$\Pi(0,1,2,4) = by^2, \quad b \in \{-1, \pm 2, \pm 3, 5, 6, \pm 10, -15, -30\}, \tag{34}$$
$$\Pi(0,1,3,4) = by^2, \quad b \in \{\pm 1, \pm 2, \pm 3, -5, 6, -15, 30\}, \tag{35}$$
$$\Pi(0,1,2,5) = by^2, \quad b \in \{-1, \pm 2, 3, \pm 5, 6, \pm 10, \pm 15\}, \tag{36}$$

correspond to the identities

$$(-3)\cdot(-1)\cdot 1 \cdot 3 = 3^2 \quad \textit{and} \quad (-2)\cdot(-1)\cdot 1 \cdot 2 = 2^2.$$

We remark that, by symmetry, results for $\Pi(0,1,2,4)$ and $\Pi(0,1,2,5)$ lead to similar statements for $\Pi(0,2,3,4)$ and $\Pi(0,3,4,5)$, respectively. Further, we may translate a claim for $\Pi(0,p,q,r)$ to one for $\Pi(i,p+i,q+i,r+i)$, for any $i \in \mathbb{Z}$.

Proof. Via the change of variables

$$X = pqb\left(\frac{rd+n}{n}\right), \quad Y = \frac{pqryb^2}{n^2},$$

if p, q and r are integers with

$$0 < p < q < r,$$

solutions in non-zero integers n, d and y to

$$\Pi(0, p, q, r) = by^2 \tag{37}$$

correspond to rational points (X, Y) on the elliptic curve

$$E : Y^2 = X(X + p(r - q)b)(X + q(r - p)b).$$

The lemma follows from the observation that, for the choices of p, q, r and b described above, the curves $E = E(p, q, r, b)$ have rank 0 over $\mathbb{Q}$ (together with a routine calculation to ensure that the torsion points yield only the stated solutions to (37)). For the given triples (p, q, r) and all other values of b dividing 30, the curves E have positive rank (and hence the equations (37) have, for these p, q, r and b, infinitely many solutions in non-zero coprime integers n and d). To verify these facts requires a routine computation in, say, *mwrank* (though *Magma* or other symbolic computation packages would be equally suitable). By way of example, if $(p, q, r) = (1, 2, 3)$, the elliptic curves corresponding to (37) are birational to the following curves listed in Table 2 (where we adopt the notation of Cremona [**9**]).

TABLE 2. *Curves birational to the elliptic curves corresponding to* (37).

b	Cremona	b	Cremona	b	Cremona	b	Cremona
1	$24A$	3	$144B$	6	$576I$	15	$3600K$
-1	$48A$	-3	$72A$	-6	$576D$	-15	$1800S$
2	$192C$	5	$600D$	10	$4800C$	30	$14400SSSS$
-2	$192D$	-5	$1200A$	-10	$4800BBB$	-30	$14400X$

If $b \in \{\pm 1, \pm 2, \pm 3, 5, -6, 15, -30\}$, then it is readily checked that the corresponding curves have rank 0. In all cases, except for $b = 1$, we have $E(\mathbb{Q})_{\text{tors}}$ isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, where the torsion points map back to only the trivial solutions to (37), with $n/d = 0, -p, -q, -r$ and $y = 0$. If $b = 1$, then there are additional torsion points given by $(X, Y) = (-2, \pm 2)$ and $(2, \pm 6)$. The latter of these corresponds to a solution to (37) with $d = 0$, while the former yields $(n, d) = (-3, 2)$.

For the remaining triples (p, q, r), we argue similarly. In all cases, for the stated values of b, we find rank 0 curves with

$$E(\mathbb{Q})_{\text{tors}} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$

unless $(p, q, r, b) = (1, 3, 4, 1)$, in which case

$$E(\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}.$$

The additional torsion points, on this model of Cremona's 48A, correspond to, again, a trivial solution to (5), and to the case $(n, d) = (-2, 1)$. □

4.2. *The case $k = 5$*

Next, let $k = 5$. By the above results for equation (33), if we write $S(m)$ for the square-free integer of maximum modulus dividing m, it follows, recalling (10), that

$$S(b_0 b_1 b_2 b_3) = S(b_1 b_2 b_3 b_4) = 6.$$

Multiplying these two terms together, we conclude that $S(b_0 b_4) = 1$ and so, since $d > 0$ implies that the sequence $\text{sign}(b_i)$ is non-decreasing in i, necessarily the b_i

are all of the same sign. On the other hand, Lemma 4.1, as applied to (35), leads to the conclusion that $S(b_0b_1b_3b_4) = -6$, a contradiction.

4.3. *The case $k = 6$*

The great majority of our work, if $l = 2$, is devoted to the situation when $k = 6$. The easy part of this case is the following result.

LEMMA 4.2. *The Diophantine equation*

$$\Pi(0, 1, 2, 3, 4, 5) = by^2, \quad b \in \{\pm 1, \pm 2, \pm 3, -5, \pm 6, \pm 10, \pm 15, 30\} \tag{38}$$

has no solutions in coprime non-zero integers n and d, with $d > 0$ and non-zero integer y.

Proof. Writing

$$n/d = (x - 5)/2, \quad y = d^3v/2^3,$$

we find that solutions to (38) correspond to rational points on the genus 2 curve

$$(x^2 - 1)(x^2 - 9)(x^2 - 25) = bv^2.$$

This genus 2 curve obviously covers the elliptic curves

$$(X - 1)(X - 9)(X - 25) = bY^2 \quad \text{and} \quad (1 - X)(1 - 9X)(1 - 25X) = bY^2.$$

It is easily checked with a suitable computer algebra package that for each of the values of b mentioned in the lemma, at least one of these curves has rank 0 and that its rational points are only the four rational 2-torsion points with $Y = 0$ or $Y = \infty$. These points correspond to solutions with $y = 0$. □

To complete the proof of Theorem 1.2 in case $k = 6$ and $l = 2$, it remains to deal with the values

$$b \in \{-30, 5\}.$$

First assume that $b = -30$. By symmetry, we may suppose that $5 \mid b_0b_1b_2$ (and consequently, $5 \nmid b_3b_4b_5$). We start with the case where $5 \mid b_0$. By Lemma 4.1 (equation (36)),

$$S(b_0b_1b_2b_5) = \pm 30 \quad \text{and} \quad S(b_0b_3b_4b_5) = \pm 30.$$

Thus $S(b_1b_2b_3b_4) = \pm 1$ which, by Lemma 4.1, gives a contradiction. If $5 \mid b_1$ then Lemma 4.1 leads to the conclusion that $S(b_2b_3b_4b_5) = 6$, whence, from the fact that $b < 0$, we have $n < 0$ and $n + d > 0$. From (34), we thus have $S(b_0b_2b_3b_4) = -6$ and $S(b_0b_1b_2b_4) = -5$ and so $S(b_0b_5) = -1$, whereby $b_0 = -1$ and $b_5 = 1$. From Lemma 4.1, as applied to (33), we thus have $b = -30$ and $S(b_1b_2b_3b_4) = 30$. It follows, then, that $b_1 = 5$ and so $S(b_2b_4) = 1$ and $b_3 = 6$, whence

$$S(b_0b_1b_2b_3) = -30,$$

contradicting Lemma 4.1. If $5 \mid b_2$ then by Lemma 4.1, as applied to (35), $S(b_0b_1b_3b_4) = -6$. As $n + 5d > 0$, we have $n + 2d > 0$. Hence $n < 0$ and $n + d > 0$. By Lemma 4.1, we have $S(b_0b_1b_2b_4) = S(b_0b_2b_3b_4) = -5$. Thus $3 \nmid b_0b_1b_3b_4$, which contradicts $S(b_0b_1b_3b_4) = -6$.

Finally, let $b = 5$. In this case, the equation $\Pi_6 = by^2$ defines a hyperelliptic curve of genus 2, which fails to cover a rank 0 elliptic curve over $\mathbb{Q}$. Further, since the Jacobian of this curve has Mordell–Weil rank 2, traditional Chabauty-type methods do not suffice to find the rational points in question. To deal with this situation, we will apply recent techniques of Bruin and Flynn [**6**] (cf. [**3**, **4**]). For our purposes, it will be preferable to consider the isomorphic curve

$$C : Y^2 = (X - 60)(X - 30)(X + 20)(X + 30)(X + 60).$$

To see how this is obtained from a solution to $\Pi_6 = 5y^2$, write $x = n/d$ and $t = 5y/d^3$, so that, after homogenizing, we have

$$t^2z^4 = 5x^6 + 75x^5z + 425x^4z^2 + 1125x^3z^3 + 1370x^2z^4 + 600xz^5.$$

The change of variables

$$x = -2X + 60Z, \quad t = -60Y, \quad z = X$$

thus leads to

$$Y^2Z^3 = X^5 + 20X^4Z - 4500X^3Z^2 - 90000X^2Z^3 + 3240000XZ^4 + 64800000Z^5$$

or, dehomogenizing, the curve C.

PROPOSITION 4.3. *The only rational solutions (X, Y) to the equation*

$$Y^2 = (X - 60)(X - 30)(X + 20)(X + 30)(X + 60)$$

are with

$$X \in \{-60, -30, -20, -15, 20, 30, 60\}.$$

Proof. Begin by observing that a rational point on C gives rise to a rational solution to the system of equations

$$\begin{aligned} X - 60 &= \delta_1 Y_1^2, \\ X - 30 &= \delta_2 Y_2^2, \\ X + 20 &= \delta_3 Y_3^2, \\ X + 30 &= \delta_4 Y_4^2, \\ X + 60 &= \delta_5 Y_5^2, \end{aligned}$$

for some 5-tuple $(\delta_1, \ldots, \delta_5)$ where $\delta_i \in \mathbb{Q}^*/\mathbb{Q}^{*2}$. In fact, since the roots of the linear factors are all distinct modulo any prime p outside the set $\{2, 3, 5\}$, it can easily be shown that these $\{\delta_i\}$ can be taken to be $\{2, 3, 5\}$-units. A straightforward 2-descent on $\mathrm{Jac}_C(\mathbb{Q})$ (see for example [**7**, **40**]) shows that the $\{\delta_i\}$ lie in a group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^6$, generated by

$$\begin{gathered} (-3, -5, 5, 15, 5),\ (3, 1, -1, -15, 5),\ (2, 5, 1, 5, 2), \\ (3, 6, 1, 15, 30),\ (15, 15, 10, 3, 30),\ (3, 1, 5, 30, 2). \end{gathered}$$

This group corresponds to the 2-Selmer group of the Jacobian of our curve. Since the torsion part of the Mordell–Weil group of $\mathrm{Jac}_C(\mathbb{Q})$ is generated by

$$\{[(60, 0) - \infty], [(30, 0) - \infty], [(-20, 0) - \infty], [(-30, 0) - \infty]\},$$

this implies, upon noting the (independent) divisors $[(-15, 3375) - \infty]$ and $[(20, 8000) - \infty]$, of infinite order, that the rank of $\mathrm{Jac}_C(\mathbb{Q})$ is 2. As mentioned

earlier, this fact ensures that a direct application of traditional Chabauty methods is not a viable option. To proceed, we will consider covers of C, as in [**3**, **4**].

Note that if the system above has a solution, then this gives rise to a solution to, say,

$$(X-60)(X-30)(X+20) = \delta_1\delta_2\delta_3(Y_1Y_2Y_3)^2.$$

Since this equation describes a genus 1 curve and there are obvious rational points on it, it models an elliptic curve, the Mordell–Weil rank of which we may bound via 2-descent. If this rank turns out to be zero, then we automatically find only a finite number of candidate solutions to our original system.

Applying this argument with all choices of three or four equations from the above system enables us to greatly reduce the possibilities for the 5-tuples $\{\delta_i\}$. We readily verify that, for the choices of $\{\delta_i\}$ which lead to coverings of rank 0 elliptic curves over $\mathbb{Q}$, the corresponding torsion points produce no points on C other than those with $Y = 0$. Carrying out this procedure for all 64 potential $\{\delta_i\}$, we see that there remain only two possible 5-tuples that lead, in all cases, to elliptic curves of positive rank, namely

$$(-3,-5,5,15,5) \text{ and } (-10,-10,10,2,5).$$

They correspond to the solutions $(X,Y) = (-15,3375)$ and $(X,Y) = (20,8000)$, respectively. This is to be expected: these are non-trivial solutions and, on each of the covered genus 1 curves, they have no particular reason to map to a torsion point. Indeed, in each case, they correspond to points of infinite order.

Note also that the original equation has an extra automorphism given by $(X,Y) \mapsto (6-X,Y)$ and that these two rational points are interchanged under this automorphism. Therefore, if we show that the values $(X,Y) = (-15,\pm 3375)$ are the only solutions corresponding to the 5-tuple $(-3,-5,5,15,5)$, then we may reach a similar conclusion, via symmetry, for $(X,Y) = (20,\pm 8000)$ and $(-10,-10,10,2,5)$. We will therefore specialize the δ_i to $(-3,-5,5,15,5)$.

From consideration of the system of equations

$$\begin{aligned} -3X+180 &= Z_1^2,\\ -5X+150 &= Z_2^2,\\ 5X+100 &= Z_3^2,\\ 15X+450 &= Z_4^2,\\ 5X+300 &= Z_5^2, \end{aligned}$$

let us therefore adopt the strategy suggested in [**6**] and analyze the fibre product of the following two covers of the X-line:

$$(-5X+150)(5X+100)(15X+450) = (Z_1Z_2Z_3)^2 \tag{39}$$

and

$$5X+300 = Z_5^2.$$

This gives us a V_4-extension of the X-line. The fibre-product D is a new curve of genus 2 with Jacobian isogenous to the product of the elliptic curve (39) and the quadratic subcover

$$(-5X+150)(5X+100)(15X+450)(5X+300) = (Z_1Z_2Z_3Z_5)^2$$

(each of these genus 1 curves has rank 1). Substituting

$$X = \left(Z_5^2 - 300\right)/5$$

into (39), we obtain a curve isomorphic to

$$D : -(u^2 - 2)(9u^2 - 8)(3u^2 - 2) = v^2.$$

Arguing as previously, we see that a rational point (u, v) on this curve gives rise to a solution of the system of equations

$$\begin{aligned} 3u^2 - 2 &= v_1^2, \\ 9u^2 - 8 &= v_2^2, \\ u^2 - 2 &= -v_3^2. \end{aligned}$$

Again, we might, via products of pairs of these equations, be led to consider elliptic covers E over $\mathbb{Q}$. The presence of the points $(\pm 1, \pm 1)$ on each of these curves, however, suggests that they will have positive rank and, indeed, it is easy to verify that they do. On the other hand, by factoring the above equations, we may obtain elliptic curves over a suitable ground field extension. This is a useful observation at this stage because, in such a situation, a rank 1 curve may still permit a successful Chabauty-type argument.

Let us choose α with $\alpha^2 = 2$ and set $K = \mathbb{Q}(\alpha)$. Consider the equations

$$Q(u) = (u - \alpha)(3u + 2\alpha)$$

and

$$R(u) = -9u^4 - 3\alpha u^3 + 18u^2 + 2\alpha u - 8.$$

Since $\mathrm{Norm}_{K/\mathbb{Q}}(Q) = (u^2 - 2)(9u^2 - 8)$, if, for $u \in \mathbb{Q}$, there are $v_1, v_2, \delta \in K^*$ satisfying

$$Q(u) = \delta v_1^2, \quad R(u) = \delta v_2^2,$$

then $-\mathrm{Norm}_{K/\mathbb{Q}}(\delta)$ must be a square in $\mathbb{Q}$. Furthermore, it is clear that δ can be taken to be a square-free $\{2, 3, 5\}$-unit in K^* (or perhaps, to be more precise, we should say a $\{\alpha, 3, 5\}$-unit).

Applying local arguments, restricting u to values in $\mathbb{Q}_p$ and seeing whether there are $v_1, v_2 \in K \otimes \mathbb{Q}_p$ satisfying the equations above, we find that, in fact, we can restrict attention to either $\delta = -\alpha - 1$ or $\delta = \alpha + 1$. These are readily seen to correspond to the points $(1, \pm 1)$ and $(-1, \pm 1)$, respectively. Again, the automorphism $(u, v) \mapsto (-u, v)$ interchanges these points. It thus suffices, by symmetry, to consider only the case where $\delta = \alpha + 1$.

We find, after a little work, that the curve defined by the equation $R(u) = (\alpha + 1)v_2^2$ is isomorphic to

$$E : y^2 = x^3 + 18(1 - \alpha)x^2 + 4(3 - 2\alpha)x.$$

In these coordinates,

$$u = \frac{(2\alpha - 3)x + (-2\alpha - 2)y - 4\alpha - 19}{7x + (2\alpha + 2)y + (-15\alpha - 36)/2}.$$

The group $E(K) = \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is generated (up to a finite index, prime to $2 \cdot 41$) by $g = (4\alpha + 6, 10\alpha + 10)$ and $T = (0, 0)$. A standard Chabauty-type argument (see [**3**, **4**]) using the prime 41 shows that 0 and $-2g$ are the only two points in $E(K)$

that yield a rational value for u, namely -1. Tracing this backwards, we find that this corresponds to the point $(X, Y) = (-15, \pm 3375)$, as claimed. This completes the proof of Proposition 4.3. □

With this proposition in place, it is a simple matter to check that the equation $\Pi_6 = 5y^2$ has only the non-trivial solutions $n \in \{-1, 6\}$ and $d = 1$, concluding the proof of Theorem 1.2 for $(k, l) = (6, 2)$.

4.4. *The cases* $k = 7,\ 8,\ 9,\ 10$ *and* 11

To treat the cases $7 \leqslant k \leqslant 11$, it is enough to observe that either there exists an i with $0 \leqslant i \leqslant k - 8$ for which $7 \mid n + id$ (so that 7 divides precisely two terms of the product Π), or that no such i exists (whence, 2 divides $\nu_7(n + jd)$ for $0 \leqslant j \leqslant k-1$). In the former case, we may apply our result for $k = 6$ to the product $\Pi(i + 1, i + 2, \ldots, i + 6)$ to reach the desired conclusion. In the latter, considering $\Pi(0, 1, 2, 3, 4, 5)$ suffices. In particular, we find only the solutions corresponding to $(n, d) = (-7, 2)$ (with $k = 8$ or 9) and to $(n, d) = (-9, 2)$ (with $k = 9$ or 10). This completes the proof of Theorem 1.2, in case $l = 2$.

5. *Proof of Theorem* 1.2 *in case* $l = 3$

As noted in § 2, given l and k, finding all coprime solutions n, d to equation (1) can be accomplished by determining the rational points on a finite number of algebraic curves. Up to this point, we have essentially relied upon equation (10) to derive single curves of, for instance, the shape (12). In this section, we will use all the information at our disposal, noting that a solution to (1), via elimination of n and d in the corresponding equations (10), implies the existence of a rational point on the non-singular curve (in $\mathbb{P}^{k-1}$) $C_{\underline{b},k,l}$, defined by the equations

$$(s - t)b_r y_r^l + (t - r)b_s y_s^l + (r - s)b_t y_t^l = 0,$$

where $\{r, s, t\}$ runs through all 3-element subsets of $\{0, \ldots, k-1\}$. Here, we write $\underline{b}$ as shorthand for $(b_0, \ldots, b_{k-1})$. We will suppress the dependence on k and l in the notation, and merely write $C_{\underline{b}}$. For the rest of this section we take $l = 3$. For any given triple $\{r, s, t\} \subset \{0, \ldots, k-1\}$, we have, as noted previously, a morphism

$$\begin{aligned} \pi_{\{r,s,t\}} : \qquad & C_{\underline{b}} \to D_{\{r,s,t\},\underline{b}}, \\ & (y_0 : \ldots : y_{k-1}) \mapsto (y_r : y_s : y_t), \end{aligned}$$

where $D_{\{r,s,t\},\underline{b}}$ is a smooth diagonal plane cubic of the form

$$D_{\{r,s,t\},\underline{b}} : Au^3 + Bv^3 + Cw^3 = 0.$$

It is convenient, for our purposes, to consider a second morphism

$$\begin{aligned} \phi : \qquad & D_{\{r,s,t\},\underline{b}} \to E_{abc}, \\ & (u : v : w) \mapsto (a^3buvw : a^3b^2v^3 : a^2w^3), \end{aligned}$$

to the curve

$$E_d \ : \ y^2z + dyz^2 = x^3.$$

Since E_d and $E_{d'}$ are isomorphic if and only if d/d' is a cube and, for our applications, we only need to consider d with $P(d) \leqslant 5$, the following lemma thus classifies the ranks of $E_d(\mathbb{Q})$ that we encounter.

LEMMA 5.1. *Let* $d = 2^{e_2}3^{e_3}5^{e_5}$ *for* $e_2, e_3, e_5 \in \{0, 1, 2\}$. *For*

$$d \in \{6, 9, 12, 15, 20, 50, 75, 90, 180, 450, 900\}$$

we have $\operatorname{rk} E_d(\mathbb{Q}) = 1$. *For other values of* d *we have* $\operatorname{rk} E_d(\mathbb{Q}) = 0$.

Proof. For each of the 27 possible values of d, the statement is easily checked with any of the computer algebra systems capable of bounding ranks of elliptic curves using 2-descent. Alternatively, one could compute the analytic ranks of these curves and, since we find them to be at most 1, conclude that they must equal the actual ranks. □

For each $C_{\underline{b}}$, it thus suffices to find an elliptic curve E_d of rank 0 which it covers. For each such rank 0 curve encountered, we may analyze each of the (finitely many) torsion points $T \in E_d(\mathbb{Q})$ and determine the rational points in the 0-dimensional fibre $(\phi \circ \pi)^{-1}(T)$. This is easily done with any modern computer algebra package; for a *Magma* [**2**] transcript of these computations, see [**5**].

We will now treat the cases $3 \leqslant k \leqslant 11$ in turn. For $3 \leqslant k \leqslant 5$ and $l = 3$, we note that Theorem 1.2 appears to be a consequence of Theorems 8 and 9 of [**19**]. Unfortunately, as we have previously noted, the proofs of these theorems require modification as they depend upon an incorrect result [**19**, Lemma 6].

5.1. *The case* $k = 3$

To begin, we need to determine the solutions to the equation

$$n(n + d)(n + 2d) = by^3,$$

for $b = 1, 2$ and 4. The coprimality of n and d implies that $\gcd(b_i, b_j) \mid (i - j)$, yielding ten possible values for $\underline{b}$.

Note that, in this case, $C_{\underline{b}}$ is the same as the curve $D_{\{0,1,2\},\underline{b}}$. Furthermore, each corresponding E_d is of rank 0. The points corresponding to the rational torsion of E_d lead, after a little work, to the arithmetic progressions (modulo reversion and $(n, d) \mapsto (-n, -d)$)

$$(-2, 1, 4), (0, 1, 2), (-1, 0, 1) \text{ and } (1, 1, 1).$$

5.2. *The case* $k = 4$

Here, we have to consider

$$b \in \{1, 2, 4, 3, 6, 12, 9, 18, 36\}.$$

Using the coprimality of n and d, these lead to 180 values of $\underline{b}$. For most choices of $\underline{b}$, one of the curves $D_{\{0,1,2\}}$, $D_{\{0,1,3\}}$, $D_{\{0,2,3\}}$ or $D_{\{1,2,3\}}$ corresponds to an E_d of rank 0. A straightforward computation shows that those values of $\underline{b}$ lead only to the arithmetic progressions

$$(0, 1, 2, 3), (-1, 0, 1, 2), (1, 1, 1, 1) \text{ and } (-3, 1, 1, 3).$$

However, for $\underline{b} = (1,2,3,4)$ or $(-6,-1,4,9)$, we find that all corresponding genus 1 subcovers of $C_{\underline{b}}$ have infinitely many rational points. Furthermore, since $(1:1:1:1) \in C_{\underline{b}}(\mathbb{Q})$, local considerations also fail to rule out these possibilities. To proceed, we need to consider other quotients of $C_{\underline{b}}$.

Let us write ζ for a primitive cube root of unity and define morphisms

$$\begin{aligned}\zeta_0 &: (y_0 : y_1 : y_2 : y_3) \mapsto (\zeta y_0 : y_1 : y_2 : y_3),\\ \zeta_1 &: (y_0 : y_1 : y_2 : y_3) \mapsto (y_0 : \zeta y_1 : y_2 : y_3),\\ \zeta_2 &: (y_0 : y_1 : y_2 : y_3) \mapsto (y_0 : y_1 : \zeta y_2 : y_3).\end{aligned}$$

Obviously, we have

$$\langle \zeta_0, \zeta_1, \zeta_2 \rangle \subset \mathrm{Aut}_{\overline{\mathbb{Q}}}(C_{\underline{b}}).$$

Writing C for one of $C_{(1,2,3,4)}$ or $C_{(-6,-1,4,9)}$, we note that quotients of C by subgroups defined over $\mathbb{Q}$ yield curves covered by C. For instance, $D_{\{1,2,3\}} \simeq C/\langle \zeta_0 \rangle$ and the corresponding E_d is isomorphic to $C/\langle \zeta_0, \zeta_1\zeta_2^2 \rangle$.

For our purposes, we will focus on the order 9 subgroup

$$H = \langle \zeta_0\zeta_1, \zeta_0\zeta_2 \rangle.$$

To derive a model for the curve $D = C/H$, we consider the H-invariant forms $y_0y_1^2y_2y_3^2$, y_2^3 and y_3^3 on C. In fact, for $\underline{b} = (1,2,3,4)$ and

$$\begin{aligned}x &= 2y_0^2y_1y_2^2y_3 / \left(9y_2^6 - 8y_2^3y_3^3\right),\\ y &= \left(-27y_2^6 + 36y_2^3y_3^3 - 16y_3^6\right) / \left(9y_2^6 - 8y_2^3y_3^3\right),\end{aligned}$$

we obtain

$$D \ : \ y^2 = x^6 - 3x^3 + 9.$$

Via a 2-descent in the style of [**8**], implemented by Stoll in *Magma* as described in [**40**], together with a point search and some canonical height computations (see [**39**, **41**]), we find that

$$\mathrm{Jac}(D)(\mathbb{Q}) \simeq \mathbb{Z}/3 \times \mathbb{Z}/3 \times \mathbb{Z}.$$

Using the identification $\mathrm{Jac}(D)(\mathbb{Q}) = \mathrm{Pic}^0(D/\mathbb{Q})$ and the convention that ∞^+ and ∞^- denote the two branches of D above $x = \infty$, we write

$$\mathrm{Jac}(D)(\mathbb{Q}) = \langle [\infty^+ - \infty^-], [(0,3) - \infty^-], [(2,7) - \infty^-] \rangle,$$

where the first two generators are of order 3 and the last generates the free part.

Via a standard application of explicit Chabauty-type methods in the style of [**15**], implemented in *Magma* by Stoll, and using $p = 7$, we compute that $D(\mathbb{Q})$ has at most six elements and that, in fact,

$$D(\mathbb{Q}) = \{\infty^+, \infty^-, (0,3), (0,-3), (2,7), (2,-7)\}.$$

When we pull back these points along the map

$$\pi : (y_0 : y_1 : y_2 : y_3) \mapsto (x, y),$$

we see that only $(2,-7)$ lifts to a rational point $(1:1:1:1) \in C(\mathbb{Q})$. This completes the first part of the proof.

For $\underline{b} = (6, 1, 4, 9)$ we proceed similarly and, in fact, writing

$$x = 2y_0y_1^2y_2y_3^2/\big(8y_2^3y_3^3 - 9y_3^6\big),$$
$$y = \big(16y_2^6 - 36y_2^3y_3^3 + 27y_3^6\big)/\big(8y_2^3y_3^3 - 9y_3^6\big),$$

find that $C_{(6,1,4,9)}$ covers the same curve D. Lifting the rational points of D along the map π yields, again, that only $(2, -7)$ gives rise to a rational point on $C_{\underline{b}}$. This completes the proof of Theorem 1.2 provided $k = 4$ and $l = 3$.

5.3. *The case* $k = 5$

If $k = 5$, dividing Π by one of n or $n + 4d$ necessarily reduces the problem to the case $k = 4$. A short calculation shows that no new solutions to (5) accrue.

5.4. *The case* $k = 6$

Let $k = 6$. If $5 \nmid (n + 2d)(n + 3d)$, then we may apply our result for $k = 4$ to $\Pi(i, i+1, i+2, i+3)$ for one of $i = 0, 1$ or 2, to conclude that the only solutions to (5), in this case, are given by

$$(n, d) = (-5, 2), (-6, 1) \text{ and } (1, 1).$$

By symmetry, we may suppose that $5 \mid n + 2d$. This leads to 1976 possible values for $\underline{b}$. For each of these, one of the twenty elliptic curves covered by $C_{\underline{b}}$ is of rank 0, whereby we can employ our previous approach. To cut down on the amount of computation required, however, it is worth noting that one can eliminate most $\underline{b}$ from consideration by testing whether $C_{\underline{b}}(\mathbb{Q}_p)$ is non-empty for, say, $p = 2, 3$ and 7. This reduces the number of $\underline{b}$ to treat to eighteen and, for each of these, $C_{\underline{b}}$ indeed has a rational point. These all correspond to the arithmetic progression

$$(-2, -1, 0, 1, 2, 3).$$

5.5. *The cases* $k = 7,\ 8,\ 9,\ 10$ *and* 11

For the cases $7 \leqslant k \leqslant 11$, we argue exactly as when $l = 2$; in all situations, consideration of one of $\Pi(i + 1, i + 2, \ldots, i + 6)$ suffices to reduce the problem to the previously treated $k = 6$. This completes the proof of Theorem 1.2 when $l = 3$.

6. *Proof of Theorem* 1.2 *in case* $l = 5$

We begin this section by proving a pair of results on ternary Diophantine equations of signature $(5, 5, 5)$. The first follows from a variety of classical arguments. The second is a consequence of work of a much more recent vintage, due to Kraus [**21**].

PROPOSITION 6.1. *Let C be a positive integer with $P(C) \leqslant 5$. If the Diophantine equation*

$$X^5 + Y^5 = CZ^5 \tag{40}$$

has solutions in non-zero coprime integers X, Y and Z, then $C = 2$ and $X = Y = \pm 1$.

Proof. Without loss of generality, we may suppose $C = 2^\alpha 3^\beta 5^\gamma$ with $0 \leqslant \alpha, \beta, \gamma \leqslant 4$. By old results of Dirichlet, Lebesgue (see for example [**13**, p. 735, item 20

and p. 738, item 37]), and P. Dénes (cf. [**12**]), for (40) to have a solution in coprime non-zero integers with $XYZ \neq \pm 1$, we require $C > 2$ and

$$C^4 \equiv 1 \pmod{5^2}.$$

This implies that $\gamma = 0$ and

$$(\alpha, \beta) \in \{(1,2),(2,4),(3,1),(4,3)\}.$$

From the fact that both 2 and 3 are primitive roots modulo 5, and the exponent 5 is a regular prime, a classical result of E. Maillet (see for example [**13**, p. 759, item 167]) leads to the conclusion that $5 \nmid Z$. Since, for each remaining value of C, we have

$$C^4 \not\equiv 2^4 \pmod{5^2},$$

Theorem 1 of Győry [**16**] thus implies that

$$r^4 \equiv 1 \pmod{5^2},$$

for every divisor r of C. The parity of the remaining C (whereby we are free to choose $r = 2$ above) provides an immediate contradiction and hence the desired result. □

PROPOSITION 6.2. *Let A and B be coprime positive integers with $AB = 2^\alpha 3^\beta$ for non-negative integers α and β with $\alpha \geqslant 4$. Then the Diophantine equation*

$$AX^5 + BY^5 = Z^5 \tag{41}$$

has no solutions in coprime non-zero integers X, Y and Z.

Proof. This is a result of Kraus [**21**] and is essentially a consequence of the fact that there are no weight 2, level N cuspidal newforms of trivial character, for N dividing 6. □

We suppose throughout this section that $l = 5$. In what follows, our arguments will typically rely upon the fact that a careful choice of identity $[q, r, s]$ leads to an equation of the form (40). In other cases, such identities imply equations which may be proven insoluble modulo 11 or 25. We shall employ the trivial observation that, for $k \leqslant 11$, at most one factor of Π_k is divisible by 11.

6.1. *The case $k = 3$*

From the identity $[0, 1, 2]$, we deduce a solution in non-zero integers to equation (40), with $P(C) \leqslant 2$ (and hence $C = 2$). A short calculation leads to the conclusion that $(n, d) = (-2, 3)$ or $(-4, 3)$.

6.2. *The case $k = 4$*

The following lemma is a more precise version of Theorem 1.2 in case $k = 4$. It will prove useful in analyzing larger values of k.

LEMMA 6.3. *Suppose that there exist non-zero integers n, d, y and b with b and d positive and $\gcd(n, d) = 1$, satisfying*

$$\Pi_4 = by^5 \quad \text{with } P(b) \leqslant 3. \tag{42}$$

Then either $(n, d) = (-3, 2)$ *or, up to symmetry,*

$$(b_0, b_1, b_2, b_3) = (4, 3, 2, 1) \text{ or } (9, 4, 1, 6). \tag{43}$$

It is likely that $(n, d) = (-9, 5), (-6, 5), (-4, 1), (-3, 2)$ and $(1, 1)$ are the only solutions of (42).

Proof. Let us suppose we have a solution to (42) in non-zero integers n, d, y and b, with $b, d > 0$ and $\gcd(n, d) = 1$. If 3 fails to divide the product $b_i b_{i+1} b_{i+2}$ for either $i = 0$ or $i = 1$, we may reduce immediately to the case $k = 3$. We may thus assume, via symmetry, that either $3 \mid b_0$ and $3 \mid b_3$, or that $3 \mid b_1$. In the first case, if $\nu_3(b_0 b_3) = 2$, the identity $[0, 1, 3]$ implies a non-trivial solution to an equation of the form $X^5 + Y^5 = 2^\alpha Z^5$ and hence, after a little work, a contradiction via Proposition 6.1. We may thus suppose, again by symmetry, that $9 \mid b_0$. Further, unless $2 \mid n + d$, we may apply the same identity $[0, 1, 3]$ to deduce a non-trivial solution to

$$X^5 + Y^5 = 2^\alpha 3^\beta Z^5, \tag{44}$$

contrary to Proposition 6.1. Combining the identities $[0, 1, 2]$ and $[0, 2, 3]$ with Proposition 6.2, we may assume that $\nu_2((n + d)(n + 3d)) = 3$. If $\nu_2(n + d) = 1$, then $[1, 2, 3]$ leads to a solution to (44) with $\alpha = \beta = 1$. If, on the other hand, $\nu_2(n + 3d) = 1$, from the fact that

$$t^5 \equiv 0, \pm 1 \pmod{11}, \quad \text{for } t \in \mathbb{Z},$$

the identity $[1, 2, 3]$ implies that $\Pi(1, 2, 3)$ is not divisible by 11. It follows from $[0, 1, 3]$ that $\nu_3(n) = 2$ (whereby (b_0, b_1, b_2, b_3) is just $(9, 4, 1, 6)$).

If, however, $3 \mid b_1$, then $[0, 1, 2]$ and Proposition 6.1 imply that we may suppose $2 \mid n$, whereby, again combining $[1, 2, 3]$, $[0, 1, 3]$ and Proposition 6.2, we may assume that $\nu_2(n(n + 2d)) = 3$. In case $\nu_2(n) = 1$, $[0, 2, 3]$ leads to a solution to (44) with $\beta = 1$, a contradiction. If $\nu_2(n) = 2$, the same identity $[0, 2, 3]$ implies that 11 fails to divide $n + 2d$ and so, modulo 11, from $[0, 1, 2]$, we are able to conclude that $\nu_3(b_1) = 1$, whence

$$(b_0, b_1, b_2, b_3) = (4, 3, 2, 1). \qquad \square$$

6.3. *The case* $k = 5$

Let $k = 5$ and suppose that we have a non-trivial solution to (5). Then applying Lemma 6.3 to $\Pi(0, 1, 2, 3)$ and $\Pi(1, 2, 3, 4)$, we see either that $n + id = -3$ and $d = 2$ for $i = 0$ or 1 (which fails to yield a solution to (5)) or that both 4-tuples (b_0, b_1, b_2, b_3) and (b_1, b_2, b_3, b_4) are in the set

$$\{(1, 2, 3, 4), (4, 3, 2, 1), (6, 1, 4, 9), (9, 4, 1, 6)\}.$$

Since this is readily seen to be impossible, we conclude that equation (5) has no solutions in this case.

6.4. *The case* $k = 6$

As in case $l = 2$ or 3, most of our work in proving Theorem 1.2 is concentrated, if $l = 5$, in treating $k = 6$. Let us suppose we have a non-trivial solution to (5) with

$P(b) \leqslant 5$. If 5 fails to divide the product $b_0b_1b_2b_3b_4b_5$, then, omitting the factor $n+5d$ in Π_6, we reduce to the case $k=5$ and hence find no new solutions.

By symmetry, it suffices to deal with the cases when $5 \mid n$, $5 \mid n+d$ or $5 \mid n+2d$. We consider them in turn.

6.4.1. *5 divides n.* First assume that $5 \mid n$ and hence also $5 \mid n+5d$. Then, applying Lemma 6.3 to $\Pi(1,2,3,4)$, we infer that either $n+d=-3$ and $d=2$ (which gives the solution $(n,d)=(-5,2)$), or we have, again up to symmetry,

$$(b_1,b_2,b_3,b_4) = (4,3,2,1) \text{ or } (9,4,1,6).$$

Consider the identity

$$3(n+d)(n+4d) - 2(n+2d)(n+3d) = n(n+5d). \tag{45}$$

If $(b_1,b_2,b_3,b_4) = (4,3,2,1)$, (45) implies a non-trivial solution to (40), contradicting Proposition 6.1. If, however, $(b_1,b_2,b_3,b_4) = (9,4,1,6)$, (45) leads to an equation of the form

$$3^4X^5 + 2^2Y^5 = 5^tZ^5$$

where $t \geqslant 2$ and $5 \nmid XY$. Working modulo 25 and taking 4th powers, we deduce the congruence

$$3^{16} \equiv 2^8 \pmod{5^2},$$

and hence a contradiction.

6.4.2. *5 divides $n+d$.* Consider now the case when $5 \mid n+d$. We apply Lemma 6.3 to $\Pi(2,3,4,5)$. It is clear that $n+2d=-3$ and $d=2$ does not provide a further solution to (5). We thus have

$$(b_2,b_3,b_4,b_5) \in \{(4,3,2,1),(1,2,3,4),(9,4,1,6),(6,1,4,9)\}.$$

In the first of these cases, necessarily $b_0 = 2\cdot 3^t$ for a non-negative integer t. From the identity $[2,3,4]$, we find that

$$2y_2^5 + y_4^5 = 3y_3^5$$

and hence 11 fails to divide $y_2y_3y_4$. Similarly, $[1,3,4]$ yields the conclusion that y_1 is coprime to 11, whereby, from $[1,2,3]$ and its companion equation

$$5^{\nu_5(b_1)}y_1^5 + 3y_3^5 = 8y_2^5,$$

we may conclude not only that $\nu_5(b_1)=1$ (so that $b_1=5$), but also

$$y_2^5 \equiv y_3^5 \equiv \pm 1 \pmod{11}.$$

Applying $[0,2,3]$, then, we obtain a solution to the equation

$$3^{\nu_3(b_0)-1}y_0^5 + y_3^5 = 2y_2^5, \tag{46}$$

and find, working modulo 11, that necessarily $\nu_3(b_0)=1$. Applying Proposition 6.1 to (46), we have $XYZ = \pm 1$. From this, we obtain the solution $(n,d)=(-6,1)$ to (5) (together with the symmetrical solution $(1,1)$).

If we have (b_2, b_3, b_4, b_5) equal to either $(1, 2, 3, 4)$ or $(9, 4, 1, 6)$, then the identities $[0, 1, 2]$ and $[0, 2, 4]$, respectively, lead to non-trivial solutions to (40), contradicting Proposition 6.1. Finally, if $(b_2, b_3, b_4, b_5) = (6, 1, 4, 9)$, then $[0, 1, 5]$ implies that

$$2^{\nu_2(b_0)+2} y_0^5 \equiv 9 y_5^5 \pmod{25}$$

and so, taking 4th powers, we conclude that $\nu_2(b_0) = 2$. This, together with $[0, 2, 4]$, contradicts Proposition 6.1.

6.4.3. 5 *divides* $n + 2d$. Finally, consider the case where $5 \mid b_2$. In light of the identity $\{0, 1, 3, 4\}$ and Proposition 3.1, we may suppose that $3 \mid n(n + d)$. First, assume that $3 \mid n$. The identity $[1, 3, 5]$ implies a non-trivial solution to (40) unless $4 \mid n + d$. Under this assumption, $[0, 3, 4]$ and Proposition 6.1 yield the conclusion that $\nu_3(n) \geqslant 2$, whence $\nu_3(n + 3d) = 1$ (and so $b_3 = 6$). From $[2, 3, 4]$, we deduce that

$$5^{\nu_5(b_2)} y_2^5 + y_4^5 = 12 y_3^5,$$

whereby, upon consideration modulo 5^2, $\nu_5(n + 2d) = 1$. Analyzing the same equation, modulo 11, implies that $11 \mid y_2$. It follows, then, from the identity $[0, 2, 4]$, that

$$3^{\nu_3(b_0)} y_0^5 + y_4^5 = 10 y_2^5.$$

Modulo 11, we therefore have $\nu_3(b_0) = 0$ and hence contradict Proposition 6.1.

The last case to consider in this subsection is when $5 \mid b_2$ and $3 \mid n + d$. From $[3, 4, 5]$ and Proposition 6.1, we may assume that $2 \mid n + d$, whence, applying a like argument with $[0, 1, 3]$, we necessarily have $\nu_2(n + d) = 1$. Identity $[0, 1, 4]$, again with Proposition 6.1, gives $\nu_3(n + 4d) \geqslant 2$ (so that $\nu_3(n + d) = 1$ and $b_1 = 6$). Applying $[0, 1, 2]$ thus leads to the equation

$$y_0^5 + 5^{\nu_5(b_2)} y_2^5 = 12 y_1^5.$$

Modulo 5^2 and 11, we again find that $\nu_5(b_2) = 1$ and that $11 \mid y_2$. To conclude, then, we apply the identity $[0, 2, 4]$ which yields

$$y_0^5 + 3^{\nu_3(b_4)} y_4^5 = 10 y_2^5.$$

This implies, modulo 11, that $\nu_3(b_4) = 0$ and so, via Proposition 6.1, a contradiction.

6.5. *The cases* $k = 7, 8, 9, 10$ *and* 11

Again, we argue as for $l = 2$ or 3, applying our results for $k = 6$ to one of $\Pi(i, i + 1, \ldots, i + 5)$. This completes the proof of Theorem 1.2.

7. *Proofs of Theorem* 1.5 *and Corollary* 1.6

Having dispatched Theorem 1.2, we will now present the proof of Theorem 1.5. The reason we proceed in this order is that the techniques introduced in this section will prove useful in the subsequent treatment of Theorem 1.4.

Proof of Theorem 1.5. If $k \leqslant 11$, Theorem 1.5 is an immediate consequence of Theorem 1.2 (without any conditions upon d). We thus assume that $k \geqslant 12$ and that $l \geqslant 2$ is prime. For the $\pi(k)$ prime values of $l \leqslant k$, we may apply Theorem 6 of [**19**] (a slight generalization of Corollary 2.1 of [**10**], itself a nice application of

Falting's Theorem) to conclude that (5) has finitely many solutions as claimed. We may thus suppose that $l > k$.

Since $d \not\equiv 0 \pmod{D_k}$ (recall definition (7)), there exists a prime in the interval $[k/2, k)$ which is coprime to d and hence divides y. Define p to be the largest such prime. From (5), since $\gcd(n, d) = 1$ and $P(b) < k/2$, it follows that either

(i) $p \mid n + id$ for precisely one i with $1 \leqslant i \leqslant k - 2$, or

(ii) $p \mid n + id$ and $p \mid n + (i + p)d$, for some i with $0 \leqslant i \leqslant k - 1 - p$.

In case (i), the identity $\{i - 1, i, i, i + 1\}$ leads to a ternary equation of the form (14) where $C = 1$ and A, B, u and v are non-zero integers with $P(AB) < p$ and $p \mid uv$. We associate to this equation, as in the proof of Proposition 3.1, a Frey elliptic curve $E/\mathbb{Q}$, with corresponding mod l Galois representation ρ_l^E. Again, this arises from a cuspidal newform f of weight 2, trivial Nebentypus character and level N. Here, from Lemma 3.2 of [**1**], N divides

$$N_1 = 64 \cdot \prod_{q<p} q,$$

where the product is over prime q. Since $p \mid uv$ and p is coprime to lN, our Frey curve E has multiplicative reduction at p and so we may conclude, as in the proof of Proposition 3.1, that

$$\mathrm{Norm}_{K_f/\mathbb{Q}}(a_p \pm (p + 1)) \equiv 0 \pmod{l},$$

where K_f is the field of definition for the Fourier coefficients a_n of f. By the Weil bounds for a_p, we have

$$l \leqslant (p + 1 + 2\sqrt{p})^{g_0^+(N)} \tag{47}$$

where $g_0^+(N)$ denotes the dimension of the space of weight 2, level N cuspidal newforms of trivial character (as a $\mathbb{C}$-vector space).

Similarly, in case (ii), we have the identity $\{i, i + j, i + p - j, i + p\}$, where we are free to choose any j with $1 \leqslant j \leqslant (p - 1)/2$. If $n(n + d)$ is odd, $p = 7$ and $k = 12$ or 13, we will take $j = 3$ whereby the above identity leads to a ternary equation of the shape (14) with coprimes A, B and C satisfying $ABC \equiv 1 \pmod 2$, $P(AB) < 7$, $C \in \{1, 3\}$ and $7 \mid uv$. Otherwise, we take $j = 2$ (if $n(n + d)$ is even) or $j = 4$ (if $n(n + d)$ is odd). These choices lead to equations (14) with $P(AB) < p$, $p - j$ divisible by C, $\gcd(AB, C) = 1$ and $p \mid uv$. Since $l > k$, in each case we may argue as previously to deduce the existence of a cuspidal newform f of weight 2, trivial Nebentypus character and level N dividing either 1440 or

$$N_2 = 64 \cdot \prod_{q_1<p} q_1 \cdot \prod_{q_2 \mid p-j} q_2$$

where again the products are over q_i prime. Arguing as before, we once more obtain inequality (47).

From Martin [**24**], we have, for any N,

$$g_0^+(N) \leqslant \frac{N + 1}{12}$$

and, via Schoenfeld [**34**],

$$\sum_{p \leqslant x} \log p < 1.000081x,$$

valid for all $x > 0$. It follows, by routine computation, that

$$g_0^+(N) < e^{1.05p}$$

and hence, from (47), that

$$\log l < 3^p < 3^k.$$

Since k is fixed, this leaves us with finitely many pairs (k, l) to consider. Again, via Theorem 6 of [**19**], we may conclude that, for each pair $(k, l) \neq (3, 2)$, equation (5) has at most finitely many coprime solutions with (6). This therefore completes the proof of Theorem 1.5. □

Proof of Corollary 1.6. To deduce Corollary 1.6, suppose now that $d \equiv 0 \pmod{D_k}$ (and, again, that $l > k$). Since it is easy to show that the left-hand side of (5) is divisible by every prime $q \leqslant k$ coprime to d, it follows, on writing

$$P_k = \pi(k-1) - \pi\left(\frac{k-1}{2}\right),$$

that

$$P_k \leqslant \omega(d) \leqslant D. \tag{48}$$

By the Prime Number Theorem, P_k is asymptotically $k/(2\log k)$, as $k \to \infty$. Applying Chebyshev-type estimates for $\pi(x)$, say those of Rosser and Schoenfeld [**29**], we may show that

$$P_k \geqslant \frac{k}{3\log k} \quad \text{if } \ k \geqslant 18.$$

From our lower bound (8) for k, we therefore have

$$P_k \geqslant \frac{2D\log D}{\log(6D\log D)} > D,$$

for $k \geqslant 18$, contradicting (48). For $12 \leqslant k \leqslant 17$ and (via inequality (8)) $D \in \{1, 2\}$, we check to see whether inequality (48) is satisfied, obtaining a contradiction in all cases except when $D = 2$ and $k = 12, 13, 15, 16$ or 17. For each of these, $P_k = 2$ and so the fact that y fails to have a prime divisor p with $k/2 \leqslant p < k$ implies that

$$d = \begin{cases} 7^\alpha 11^\beta & \text{if } k = 12, 13, \\ 11^\alpha 13^\beta & \text{if } k = 15, 16, 17, \end{cases}$$

where α and β are positive integers. Theorem 2 of Saradha and Shorey [**32**], however, shows that d necessarily has a prime divisor congruent to 1 (mod l). It follows that $l \in \{2, 3, 5\}$, contradicting $l > k$. This completes the proof of Corollary 1.6. □

8. *Finiteness results for* $12 \leqslant k \leqslant 82$

In this section, we will present the proof of Theorem 1.4. We begin by noting that if P and Q are consecutive primes and if we know that equation (5) has finitely many solutions with $k = 2P + 1$ and (6), then a similar result is immediately obtained for

$$k = 2P + 2, 2P + 3, \ldots, 2Q.$$

Indeed, for any of these values of k, if Π_k is divisible by a prime in the interval $[Q, k]$, then Theorem 1.5 implies the desired result. We may thus suppose, if $p \mid \Pi_k$, that either $p > k$ or $p \leqslant P$. It follows that we can write

$$\Pi(0, 1, \dots, 2P+1) = BY^l$$

for non-zero integers B and Y with $P(B) \leqslant P$, whereby the result follows, as claimed, from the case $k = 2P + 1$. To prove Theorem 1.4, we may, in light of Theorem 1.2, restrict attention to

$$k \in \{15, 23, 27, 35, 39, 47, 59, 63, 75\},$$

where we further suppose that Π_k is coprime to D_k. Now, for each prime $3 \leqslant p \leqslant P$, there are $p + 1$ possibilities: either $p \mid n + sd$ for some $0 \leqslant s \leqslant p - 1$, or p fails to divide Π (that is, $p \mid d$). Analyzing these

$$N(P) = \prod_{3 \leqslant p \leqslant P} (p + 1) \tag{49}$$

cases, for each k under consideration (actually, symmetry allows us to reduce this number somewhat), we note that if we can find integers $i \geqslant 0$ and $j \geqslant 1$ such that $6j + i \leqslant k - 1$ and

$$\gcd\left(\Pi(i, 3j + i, 6j + i), \prod_{3 \leqslant p \leqslant P} p\right) \in \{1, 11, 19\}, \tag{50}$$

then $\{i, 3j + i, 3j + i, 6j + i\}$ leads to an equation of the form (22). We obtain a similar conclusion if there exist $i \geqslant 0$ and $j \geqslant 1$ with $10j + i \leqslant k - 1$, for which

$$\gcd\left(\Pi(i, j + i, 9j + i, 10j + i), \prod_{3 \leqslant p \leqslant P} p\right) \in \{1, 11, 19\} \tag{51}$$

(where we employ the identity $\{i, j + i, 9j + i, 10j + i\}$).

8.1. *The case $k = 15$*

For $k = 15$ (that is, if $P = 7$), a short search indicates that we can find i and j for which (50) or (51) holds, unless $p \mid n + i_p d$ for $p \in \{3, 5, 7\}$ where i_p are as shown in Table 3.

TABLE 3.

Case	(i_3, i_5, i_7)	Case	(i_3, i_5, i_7)	Case	(i_3, i_5, i_7)
(i)	(2, 4, 6)	(v)	(0, 3, 4)	(ix)	(2, 4, 0)
(ii)	(1, 3, 5)	(vi)	(2, 2, 3)	(x)	(1, 3, 6)
(iii)	(0, 2, 4)	(vii)	(1, 1, 2)	(xi)	(1, 1, 1)
(iv)	(2, 1, 3)	(viii)	(0, 0, 1)	(xii)	(0, 0, 0)

By symmetry, we may suppose that we are in one of the cases (i), (ii), (iii), (iv), (ix) or (x). In case (i), $\{1, 3, 10, 12\}$ implies an equation of the form (18) with $D = 2$ if Π is odd, and (15) with $\beta = 0$ if Π is even, unless, in this latter case, we have

$$\max\{\nu_2(n + id) : i = 1, 3, 10, 12\} = 2. \tag{52}$$

It follows, in this situation, that $\{2,3,11,12\}$ leads to equation (15) with $\alpha \geqslant 2$, unless $9 \mid n+2d$. If we assume, then, that $9 \mid n+2d$, $\{5,7,8,10\}$ implies an equation of the form (15) with $\beta = 0$, unless

$$\max\{\nu_2(n+id) : i = 5,7,8,10\} = 2. \tag{53}$$

Combining (52) and (53), we may thus assume that $\nu_2(n+10d) = 2$, whereby $\{5,8,9,12\}$ leads to an equation of the form (20), completing the proof in case (i).

In cases (ii), (ix) and (x), we argue in an identical fashion as for case (i), only with the identities

$$\{1,3,10,12\},\ \{2,3,11,12\},\ \{5,7,8,10\} \text{ and } \{5,8,9,12\}$$

replaced by

$$\{0,2,9,11\},\ \{1,2,10,11\},\ \{4,6,7,9\},\ \{4,7,8,11\},\ \text{in case (ii)},$$
$$\{1,3,10,12\},\ \{2,3,11,12\},\ \{3,5,6,8\},\ \{1,4,5,8\},\ \text{in case (ix)}$$

and

$$\{0,2,9,11\},\ \{1,2,10,11\},\ \{2,4,5,7\},\ \{0,3,4,7\},\ \text{in case (x)}.$$

In case (iii) (respectively case (iv)), the identity $\{1,5,10,14\}$ (respectively $\{0,4,9,13\}$) leads to the conclusion that

$$\max\{\nu_2(n+id) : i = 1,5,10,14\} = 3$$

whence $\{8,9,9,10\}$, $\{2,5,5,8\}$, $\{7,10,10,13\}$ and $\{3,6,6,9\}$ (respectively $\{7,8,8,9\}$, $\{1,4,4,7\}$, $\{6,9,9,12\}$ and $\{2,5,5,8\}$) lead to equations of the shape (21) with $p \in \{3,5\}$. This completes the proof of Theorem 1.4 if $k = 15$ (that is, for $k \leqslant 22$).

8.2. *The cases $k \in \{23,27,35,39\}$*

A (reasonably) short calculation reveals that for each of the $N(P)$ possibilities with $P \in \{11,13,17\}$, we can always find i and j satisfying (50) or (51). If $P = 19$ (so that $k = 39$), then we are left with, up to symmetry, the cases listed in Table 4 to consider (where, as previously, $p \mid n + i_p d$).

TABLE 4.

Case	i_3	i_5	i_7	i_{11}	i_{13}	i_{17}	i_{19}
(i)	1	0	3	6	1	9	6
(ii)	1	0	4	1	8	9	17
(iii)	0	4	3	0	7	8	16
(iv)	2	3	2	10	6	7	15

In the first of these $\{8,11,33,36\}$ leads immediately to an equation of the shape (15) with $\beta = 1$. In the remaining three,

$$\{2-i, 6-i, 29-i, 33-i\}$$

(for $i = 0, 1$ or 2, respectively) implies a solution to equation (15) with $(\alpha,\beta) = (0,1)$, if Π is odd. If, however, Π is even, the identity

$$\{28-i, 29-i, 37-i, 38-i\}$$

leads to equation (15) with $\alpha \geqslant 2$ and $\beta = 1$, unless $9 \mid n + (28 - i)d$. In this case, the identity

$$\{13 - i, 14 - i, 16 - i, 17 - i\}$$

thus leads to equation (22) with $p = 19$. This completes the proof for $k = 39$ (and hence for $k \leqslant 46$).

8.3. *The cases* $k \in \{47, 59, 63, 75\}$

We verify via *Maple* that, for each of the $N(P)$ possibilities with $P \in \{23, 29\}$, we can always find i and j satisfying (50) or (51). For $P = 31$ (that is, $k = 63$), there are again some possibilities that elude our sieve (the computation is now becoming rather more substantial). These 28 cases correspond, after symmetry, to $p \mid n + i_p d$ for i_p as shown in Table 5.

TABLE 5.

Case	i_3	i_5	i_7	i_{11}	i_{13}	i_{17}	i_{19}	i_{23}	i_{29}	i_{31}
(i)	0	3	5	1	7	1	18	2	14	10
(ii)	2	2	4	0	6	0	17	1	13	9
(iii)	0	3	5	1	7	15	18	14	16	10
(iv)	2	2	4	0	6	14	17	13	15	9
(v)	1	1	3	8	5	15	11	4	8	23
(vi)	0	0	2	7	4	14	9	3	7	22
(vii)	2	4	1	6	3	13	8	2	6	21
(viii)	1	1	3	8	5	15	8	4	1	23
(ix)	0	3	5	10	7	14	13	6	10	25
(x)	2	2	4	9	6	13	12	5	9	24
(xi)	0	0	2	3	4	14	9	3	7	22
(xii)	2	4	1	2	3	13	8	2	6	21
(xiii)	0	3	5	10	7	14	10	6	3	25
(xiv)	2	2	4	9	6	13	9	5	2	24

Our arguments will prove similar in each case. From an initial identity of the form $\{p, q, r, s\}$, we will conclude that $8 \mid n + id$ for some i congruent, modulo 8, to $p+4$, $q+4$, $r+4$ or $s+4$. For each of these possibilities, one of a collection of 4 (or 2) secondary identities of the shape $\{p_1, q_1, q_1, r_1\}$ then implies a non-trivial solution to an equation of the form (21), contradicting Proposition 3.1. For example, in case (i), $\{31, 32, 49, 50\}$ implies the desired conclusion unless

$$\max\{\nu_2(n + id) : i = 31, 32, 49, 50\} = 2.$$

This hypothesis ensures that $8 \mid n + id$ for one of $i = 3, 4, 5, 6$ which, with the identities $\{6, 11, 11, 16\}$, $\{11, 20, 20, 29\}$, $\{4, 13, 13, 22\}$ and $\{29, 30, 30, 31\}$, contradicts Proposition 3.1. For the remaining cases, we choose our identities as indicated in Tables 6 and 7.

TABLE 6.

Case	Initial identity	$8 \mid n + id$
(i)	$\{31, 32, 49, 50\}$	$i = 3, 4, 5, 6$
(iii)	$\{2, 4, 29, 31\}$	$i = 0, 2, 3, 5$
(v), (viii)	$\{12, 14, 60, 62\}$	$i = 4, 6$
(ix), (xiii)	$\{16, 17, 34, 35\}$	$i = 4, 5, 6, 7$
(xi)	$\{11, 13, 59, 61\}$	$i = 3, 5$

TABLE 7.

Case	Secondary identities
(i)	$\{6, 11, 11, 16\}, \{11, 20, 20, 29\}, \{4, 13, 13, 22\}, \{29, 30, 30, 31\}$
(iii)	$\{21, 24, 24, 27\}, \{27, 42, 42, 57\}, \{2, 11, 11, 20\}, \{4, 13, 13, 22\}$
(v), (viii)	$\{13, 28, 28, 43\}, \{9, 22, 22, 35\}$
(ix), (xiii)	$\{1, 4, 4, 7\}, \{16, 37, 37, 58\}, \{17, 22, 22, 27\}, \{2, 23, 23, 44\}$
(xi)	$\{6, 19, 19, 32\}, \{8, 13, 13, 18\}$

In case (ii), (iv), (vi), (vii), (x), (xii) and (xiv), we argue as for (i), (iii), (v), (ix) and (xi), but with $\{p, q, r, s\}$ replaced, in each case, by $\{p - i, q - i, r - i, s - i\}$ for $i = 1$ or $i = 2$. This completes the proof of Theorem 1.4, for $63 \leqslant k \leqslant 74$.

To finish the proof of Theorem 1.4, it remains to handle the case $k = 75$. In this situation, after lengthy calculations (carried out in *Maple* on a Beowulf cluster at Simon Fraser University), we conclude that there always exist i and j satisfying either (50) or (51). The code utilized in this computation is available from the authors on request.

9. *Concluding remarks*

Presumably, the cases $2 \leqslant l \leqslant 5$ in Theorem 1.2 may be sharpened with a more careful combinatorial analysis, at least if $(k, l) \neq (4, 2)$ or $(3, 3)$. As far as we can tell, the statement, for large prime values of l, essentially reflects the limitations of our method. An extension of Theorem 1.2 to larger values of k would be a reasonably routine matter if one had available a full set of Galois conjugacy classes of weight 2 cuspidal newforms at larger levels than currently present in [**38**]. Proving an analog of Theorem 1.4 for larger k is also certainly possible via the techniques described herein; to some degree, at this stage, the problem is primarily a matter of combinatorics.

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